#1: if $f(x) = x$ then $b_n = \dfrac{2(-1)^{n+1}}{n}$

If, in addition $c = -1$ then we have

$$(-1-n^2)\, c_n = b_n = \frac{2(-1)^{n+1}}{n}$$

$$c_n = \frac{2(-1)^{n+1}}{-1(1+n^2)n}$$

$$c_n = \frac{2(-1)^n}{(1+n^2)n}$$

the exact solution to this problem, by undetermined coefficients

$$y'' - y = x \qquad 0 < x < \pi$$
$$y(0) = y(\pi) = 0$$

$$y_c = c_1 e^x + c_2 e^{-x}$$

$$Y = Ax + B$$
$$Y'' = 0 \quad \text{so} \quad Y'' - Y = -Ax - B = x$$
$$\Rightarrow B = 0, \quad x = -1$$

$$y = c_1 e^x + c_2 e^{-x} - x$$
$$y(0) = c_1 + c_2 - 0 = 0 \qquad \Rightarrow \quad c_2 = -c_1$$
$$y(\pi) = c_1 e^{\pi} + c_2 e^{-\pi} - \pi = 0$$

$$c_1 e^{\pi} - c_1 e^{-\pi} - \pi = 0$$
$$c_1 (e^{\pi} - e^{-\pi}) = \pi$$
$$2\sinh\pi \cdot c_1 = \pi$$
$$c_1 = \frac{\pi}{2\sinh\pi}$$

(If you don't remember that $\sinh x = \dfrac{e^x - e^{-x}}{2}$ that's okay.)

$$y = \frac{\pi}{2\sinh\pi}\left(e^{x} - e^{-x}\right) - x$$

$$= \frac{\pi}{2\sinh\pi} \cdot 2\sinh x - x$$

$$\boxed{y = \frac{\pi}{\sinh\pi}\sinh x - x}$$

Finally if $c = 1$ then

$(1-n^2)\,C_n = b_n$ if $n=1$ & $b_1 \neq 0$ then

this gives $0 \cdot C_1 = b_1 \neq 0$ for $n=1$, which can't be solved, but if $b_1 = 0$ then the equation is satisfied for any c_1

This is yet another example of

Exercise 2

Example 1 $\quad y'' = f(x) \quad 0 < x < \pi$

$y'(0) = y'(\pi) = 0$

Assume $\quad y = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos nx \quad$ (coefficients as yet unknown)

and $\quad f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

where $\quad a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx \, dx$

Then $\quad y'' = \sum_{n=1}^{\infty} -n^2 c_n \cos nx$

Setting $\quad y'' = f \Rightarrow \sum_{n=1}^{\infty} -n^2 c_n \cos nx = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

This can be solved only if $\frac{a_0}{2} = 0$, in which case

c_0 can take any value and

$c_n = \frac{-a_n}{n^2}$ for $n > 0$

Finally let's look at

$$y'' + cy = f(x) \qquad 0 < x < \pi, \quad c \neq 0$$

then we get

$$y'' + cy = \frac{c \cdot c_0}{2} + \sum_{n=1}^{\infty} (c - n^2) c_n$$

setting this equal to f gives

$$\frac{c \cdot c_0}{2} = \frac{a_0}{2} \implies c_0 = \frac{a_0}{c}$$

$$(c - n^2) c_n = a_n \implies c_n = \frac{a_n}{c - n^2}$$ again assuming $n^2 \neq c$