SUPPLEMENTARY MATERIAL FOR: A NEW CANONICAL REDUCTION OF THREE-VORTEX MOTION AND ITS APPLICATION TO VORTEX-DIPOLE SCATTERING

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Note: Equation and figure numbers refer to numbering in the published paper. Local equation and figure labels begin with the letters "SM."

In this supplement, calculate the change in angle $\Delta\alpha$ on trajectories with initial conditions as $t \to -\infty$ given in Fig. 6. The result is equivalent to one calculated in the supplementary material to [3]. We include it for completeness and to highlight the connection with the phase planes of Fig. 8.

To obtain an explicit integral form, we divide $\frac{\mathrm{d}\alpha}{\mathrm{d}t}$ from Eq. (30) by $\frac{\mathrm{d}Y}{\mathrm{d}t}$, given by Eq. (28b), yielding $\frac{\mathrm{d}\alpha}{\mathrm{d}Y}$. We remove the dependence on X and Z using the conservation laws (27) and (25), and then replace H by its value given the initial condition in Fig. 6. We will use Θ instead of ρ as the parameter in what follows because it gives somewhat simpler formulas and can use Eq. (33) to rewrite this in terms of the parameter ρ defining the initial conditions. Integrating this, we find

$$\Delta\alpha = \int_{Y_{\min}}^{\infty} \frac{-8\Theta^2 \mathrm{d}Y}{(Y^2+\Theta^2)\sqrt{p_4(Y^2;\Theta)}} + \int_{Y_{\min}}^{\infty} \frac{8(\Theta^2-8\Theta)\mathrm{d}Y}{(Y^2+\Theta^2-8\Theta)\sqrt{p_4(Y^2;\Theta)}} \tag{SM1}$$

where

$$p_4(Y^2;\Theta) = Y^4 + 2\left(\Theta^2 - 4\Theta - 8\right)Y^2 + (\Theta-8)\Theta^3.$$

These are *complete elliptic integrals* [1]. To place them in standard form, we must first factor $p_4(Y^2;\Theta)$. We plot its zero locus in Fig. SM1 as a function of Θ and Y^2. From this image, it is clear that p_4 can be factored as follows

$$p_4(Y^2,\Theta) = \begin{cases} (Y^2-(a+ib)^2)(Y^2-(a-ib)^2), & a>0, b>0, \text{ if } \Theta < -1; \\ (Y^2-a^2)(Y^2-b^2), & a>b>0, \text{ if } -1<\Theta<0; \\ (Y^2-a^2)(Y^2+b^2), & a>0, b>0, \text{ if } 0<\Theta<8; \\ (Y^2+a^2)(Y^2+b^2), & a>b>0, \text{ if } 8<\Theta. \end{cases} \tag{SM2}$$

The first two cases correspond to the left phase plane of Fig. 8, the last two to the right phase plane; the first and last cases correspond to direct scattering, and the second and third to exchange scattering. The lower limit of integration is $Y_{\min} = 0$ in the first and fourth cases, while in the second and third $Y_{\min} = a$. Both integrals in Eq. (SM1) can be evaluated with the help of references such as Gradshteyn/Ryzhik and Byrd/Friedman[1, 2]. It is quite possible that these expressions can be simplified further. For example, Lydon derived formulas in which α is the sum of one complete elliptic integral of the first kind and one of the third kind.

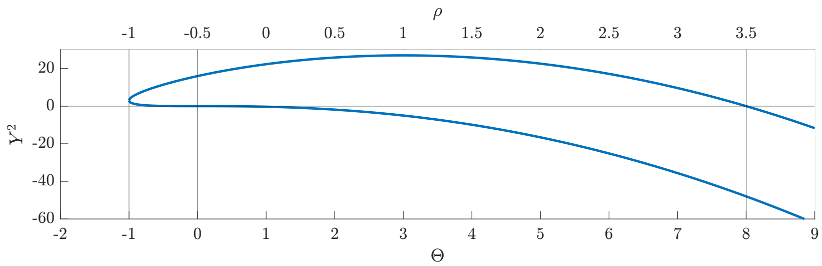

FIGURE SM1. The solutions to $p_4(Y^2,\Theta)=0$, with the transitions between the factored form in Eq. (SM2) marked be vertical lines.

In the four regions, the constants evaluate to the following

$$\begin{pmatrix} a^2 \\ b^2 \end{pmatrix} = \begin{cases} \frac{1}{2}\begin{pmatrix} \sqrt{\Theta-8}\Theta^{3/2}-\Theta^2+4\Theta+8 \\ \sqrt{\Theta-8}\Theta^{3/2}+\Theta^2-4\Theta-8 \end{pmatrix} & \text{if } \Theta<-1; \\ \begin{pmatrix} -\Theta^2+4\Theta+8\sqrt{\Theta+1}+8 \\ -\Theta^2+4\Theta-8\sqrt{\Theta+1}+8 \end{pmatrix} & \text{if } -1<\Theta<0; \\ \begin{pmatrix} -\Theta^2+4\Theta+8\sqrt{\Theta+1}+8 \\ \Theta^2-4\Theta+8\sqrt{\Theta+1}-8 \end{pmatrix} & \text{if } 0<\Theta<8; \\ \begin{pmatrix} \Theta^2-4\Theta+8\sqrt{\Theta+1}-8 \\ \Theta^2-4\Theta-8\sqrt{\Theta+1}-8 \end{pmatrix} & \text{if } 8<\Theta. \end{cases}$$

In each of the four ρ intervals, the scattering angle can be written as a linear combination of complete elliptic integrals of the first kind

$$K(m)=\int_0^1 \frac{\mathrm{d}x}{\sqrt{(1-x^2)(1-mx^2)}}=\int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\theta}{\sqrt{1-m\sin^2\theta}}.$$

and the third kind

$$\Pi(n,m)=\int_0^1 \frac{\mathrm{d}x}{(1-nx^2)\sqrt{(1-x^2)(1-mx^2)}}=\int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\theta}{\left(1-n\sin^2\theta\right)\sqrt{1-m\sin^2\theta}}.$$

The convention is to define these functions for $0<m<1$, though they are analytic for all m except for a branch cut from $m=1$ to $m=\infty$.

We report the values found in each of the cases.

DIRECT SCATTERING WITH $\rho<-1$

Here $\Theta<-1$, and

$$\Delta\alpha=\frac{64\sqrt[4]{\Theta-8}\,(-K(m)+\Pi(n,m))}{\sqrt[4]{\Theta}\left(\Theta-8+\sqrt{\Theta^2-8\Theta}\right)\left(\Theta+\sqrt{\Theta^2-8\Theta}\right)}$$

with

$$m=\frac{1}{2}+\frac{4-\Theta}{2\sqrt{\Theta^2-8\Theta}} \quad \text{and} \quad n=\frac{1}{2}-\frac{\Theta^2-4\Theta-8}{2\Theta\sqrt{\Theta^2-8\Theta}}.$$

EXCHANGE SCATTERING WITH $-1 < \rho < -\frac{1}{2}$

In this case $-1 < \Theta < 0$, and

$$\Delta\alpha = \frac{4\Theta K(m) + 8\sqrt{1+\Theta}\Big(\Pi(n_1,m) - \Pi(n_2,m)\Big)}{\sqrt{-\Theta^2 + 4\Theta + 8\sqrt{\Theta+1} + 8}},$$

where

$$\begin{aligned} m &= \frac{8 + 4\Theta - \Theta^2 - 8\sqrt{\Theta+1}}{8 + 4\Theta - \Theta^2 + 8\sqrt{\Theta+1}}, \\ n_1 &= \frac{\Theta - 2 + 2\sqrt{\Theta+1}}{\Theta + 2 - 2\sqrt{\Theta+1}}, \\ \text{and} \quad n_2 &= \frac{\Theta + 2 - 2\sqrt{\Theta+1}}{\Theta + 2 + 2\sqrt{\Theta+1}}. \end{aligned}$$

THE BORDERLINE CASE $\rho = -\frac{1}{2}$

This is the case $\Theta = 0$ discussed in Fig. 9. Vortex 2 travels along a straight line with no deflection, so the scattering angle is $\alpha = 0$.

EXCHANGE SCATTERING WITH $-\frac{1}{2} < \rho < \frac{7}{2}$

Here $0 < \Theta < 8$, and

$$\Delta\alpha = \frac{-\Theta^2 + 4\Theta + 8\sqrt{\Theta+1} + 8}{2\sqrt[4]{\Theta+1}\left(\Theta + 2\sqrt{\Theta+1} + 2\right)}\Big(\Pi(n_1,m) - \Pi(n_2,m)\Big),$$

where

$$\begin{aligned} m &= \frac{1}{2} + \frac{\Theta^2 - 4\Theta - 8}{16\sqrt{1-\Theta}}; \\ n_1 &= \frac{2 - \Theta - 2\sqrt{1+\Theta}}{4}; \\ \text{and} \quad n_2 &= \frac{2 + \Theta - 2\sqrt{1+\Theta}}{4}. \end{aligned}$$

DIRECT SCATTERING WITH $\frac{7}{2} < \rho$

In this last case, $\Theta > 8$ and

$$\Delta\alpha = c_K K(m) + c_{\Pi,1}\Pi_1(n_1,m) + c_{\Pi,2}\Pi(n_2,m),$$

where

$$m = \frac{16\sqrt{\Theta+1}}{\Theta^2 - 4\Theta - 8 + 8\sqrt{\Theta+1}}, n_1 = -\frac{4}{\Theta + 2\sqrt{\Theta+1} - 2}, n_2 = \frac{4\left(\Theta + 2\sqrt{\Theta+1} + 2\right)}{\Theta^2}$$

$$c_K = -\frac{4\Theta}{\sqrt{\Theta^2 - 4\Theta + 8\sqrt{\Theta+1} - 8}}, c_{\Pi,1} = \frac{-2\Theta^3 + 4\Theta^2 + 64\Theta + 64 - 4\sqrt{\Theta+1}\left(\Theta^2 - 8\Theta - 16\right)}{\sqrt{(\Theta-8)\Theta^3\left((\Theta-4)\Theta - 8\left(\sqrt{\Theta+1} + 1\right)\right)}},$$

$$\text{and} \quad c_{\Pi,2} = \frac{-2\Theta^3 + 12\Theta^2 - 32\Theta - 64 + 4\left(\Theta^2 - 16\right)\sqrt{\Theta+1}}{\sqrt{(\Theta-8)\Theta^3\left((\Theta-4)\Theta - 8\left(\sqrt{\Theta+1} + 1\right)\right)}}$$

REFERENCES

[1] P. Byrd and M. Friedman. *Handbook of Elliptic Integrals for Engineers and Scientists*. Grundlehren der mathematischen Wissenschaften. Springer Berlin Heidelberg, 2nd edition, 1971.

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[3] K. Lydon, S. V. Nazarenko, and J. Laurie. Dipole dynamics in the point vortex model. *J. Phys. A: Math. Theor.*, 55:385702, 2022.

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