Note: Equation and figure numbers refer to numbering in the published paper. Local equation and figure labels begin with the letters “SM.”

In this supplement, calculate the change in angle $\Delta \alpha$ on trajectories with initial conditions as $t \to -\infty$ given in Fig. 6. The result is equivalent to one calculated in the supplementary material to [3]. We include it for completeness and to highlight the connection with the phase planes of Fig. 8.

To obtain an explicit integral form, we divide $\frac{d\alpha}{dt}$ from Eq. (30) by $\frac{dY}{dt}$, given by Eq. (28b), yielding $\frac{d\alpha}{dY}$. We remove the dependence on $X$ and $Z$ using the conservation laws (27) and (25), and then replace $H$ by its value given the initial condition in Fig. 6. We will use $\Theta$ instead of $\rho$ as the parameter in what follows because it gives somewhat simpler formulas and can use Eq. (33) to rewrite this in terms of the parameter $\rho$ defining the initial conditions. Integrating this, we find

\[ (SM1) \quad \Delta \alpha = \int_{Y_{\min}}^\infty \frac{-8\Theta^2 dY}{(Y^2 + \Theta^2)\sqrt{p_4(Y^2; \Theta)}} + \int_{Y_{\min}}^\infty \frac{8(\Theta^2 - 8\Theta)dY}{(Y^2 + \Theta^2 - 8\Theta)\sqrt{p_4(Y^2; \Theta)}} \]

where

\[ p_4(Y^2; \Theta) = Y^4 + 2(\Theta^2 - 4\Theta - 8)Y^2 + (\Theta - 8)\Theta^3. \]

These are complete elliptic integrals [1]. To place them in standard form, we must first factor $p_4(Y^2; \Theta)$. We plot its zero locus in Fig. SM1 as a function of $\Theta$ and $Y^2$. From this image, it is clear that $p_4$ can be factored as follows

\[ (SM2) \quad p_4(Y^2, \Theta) = \begin{cases} 
(Y^2 - (a + ib)^2)(Y^2 - (a - ib)^2), & a > 0, b > 0, \text{ if } \Theta < -1; \\
(Y^2 - a^2)(Y^2 - b^2), & a > b > 0, \text{ if } -1 < \Theta < 0; \\
(Y^2 - a^2)(Y^2 + b^2), & a > 0, b > 0, \text{ if } 0 < \Theta < 8; \\
(Y^2 + a^2)(Y^2 + b^2), & a > b > 0, \text{ if } 8 < \Theta.
\end{cases} \]

The first two cases correspond to the left phase plane of Fig. 8, the last two to the right phase plane; the first and last cases correspond to direct scattering, and the second and third to exchange scattering. The lower limit of integration is $Y_{\min} = 0$ in the first and fourth cases, while in the second and third $Y_{\min} = a$. Both integrals in Eq. (SM1) can be evaluated with the help of references such as Gradshteyn/Ryzhik and Byrd/Friedman[1, 2]. It is quite possible that these expressions can be simplified further. For example, Lydon derived formulas in which $\alpha$ is the sum of one complete elliptic integral of the first kind and one of the third kind.
In the four regions, the constants evaluate to the following

\[
\begin{cases} 
\frac{\sqrt{\Theta - 8} \Theta^{3/2} - \Theta^2 + 4\Theta + 8}{\sqrt{\Theta - 8} \Theta^{3/2} + \Theta^2 - 4\Theta - 8} & \text{if } \Theta < -1; \\
-\Theta^2 + 4\Theta + 8\sqrt{\Theta + 1} + 8 & \text{if } -1 < \Theta < 0; \\
-\Theta^2 + 4\Theta - 8\sqrt{\Theta + 1} + 8 & \text{if } 0 < \Theta < 8; \\
\Theta^2 - 4\Theta + 8\sqrt{\Theta + 1} - 8 & \text{if } 8 < \Theta.
\end{cases}
\]

In each of the four \( \rho \) intervals, the scattering angle can be written as a linear combination of complete elliptic integrals of the first kind

\[
K(m) = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-mx^2)}} = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-m\sin^2 \theta}}.
\]

and the third kind

\[
\Pi(n, m) = \int_0^1 \frac{dx}{(1-nx^2)\sqrt{(1-x^2)(1-mx^2)}} = \int_0^{\frac{\pi}{2}} \frac{d\theta}{(1-n\sin^2 \theta)\sqrt{1-m\sin^2 \theta}}.
\]

The convention is to define these functions for \( 0 < m < 1 \), though they are analytic for all \( m \) except for a branch cut from \( m = 1 \) to \( m = \infty \).

We report the values found in each of the cases.

**Direct scattering with \( \rho < -1 \)**

Here \( \Theta < -1 \), and

\[
\Delta \alpha = \frac{64\sqrt{\Theta - 8}(-K(m) + \Pi(n, m))}{\sqrt{\Theta}(\Theta - 8 + \sqrt{\Theta^2 - 8\Theta})(\Theta + \sqrt{\Theta^2 - 8\Theta})}
\]

with

\[
m = \frac{1}{2} + \frac{4 - \Theta}{2\sqrt{\Theta^2 - 8\Theta}} \quad \text{and} \quad n = \frac{1}{2} - \frac{\Theta^2 - 4\Theta - 8}{2\Theta\sqrt{\Theta^2 - 8\Theta}}.
\]
EXCHANGE SCATTERING WITH $-1 < \rho < -\frac{1}{2}$

In this case $-1 < \Theta < 0$, and

$$\Delta \alpha = \frac{4\Theta K(m) + 8\sqrt{1+\Theta} \left( \Pi(n_1,m) - \Pi(n_2,m) \right)}{\sqrt{-\Theta^2 + 4\Theta + 8\sqrt{\Theta + 1} + 8}},$$

where

$$m = \frac{8 + 4\Theta - \Theta^2 - 8\sqrt{\Theta + 1}}{8 + 4\Theta - \Theta^2 + 8\sqrt{\Theta + 1}},$$

$$n_1 = \frac{\Theta - 2 + 2\sqrt{\Theta + 1}}{\Theta + 2 - 2\sqrt{\Theta + 1}},$$

and

$$n_2 = \frac{\Theta + 2 - 2\sqrt{\Theta + 1}}{\Theta + 2 + 2\sqrt{\Theta + 1}}.$$

THE BORDERLINE CASE $\rho = -\frac{1}{2}$

This is the case $\Theta = 0$ discussed in Fig. 9. Vortex 2 travels along a straight line with no deflection, so the scattering angle is $\alpha = 0$.

EXCHANGE SCATTERING WITH $-\frac{1}{2} < \rho < \frac{7}{2}$

Here $0 < \Theta < 8$, and

$$\Delta \alpha = \frac{-\Theta^2 + 4\Theta + 8\sqrt{\Theta + 1} + 8}{2\sqrt{\Theta + 1} (\Theta + 2\sqrt{\Theta + 1} + 2)} \left( \Pi(n_1,m) - \Pi(n_2,m) \right),$$

where

$$m = \frac{1}{2} + \frac{\Theta^2 - 4\Theta - 8}{16\sqrt{1-\Theta}};$$

$$n_1 = \frac{2 - \Theta - 2\sqrt{1+\Theta}}{4};$$

and

$$n_2 = \frac{2 + \Theta - 2\sqrt{1+\Theta}}{4}.$$

DIRECT SCATTERING WITH $\frac{7}{2} < \rho$

In this last case, $\Theta > 8$ and

$$\Delta \alpha = c_K K(m) + c_{\Pi,1} \Pi_1(n_1,m) + c_{\Pi,2} \Pi(n_2,m),$$

where

$$c_K = -\frac{4\Theta}{\sqrt{\Theta^2 - 4\Theta + 8\sqrt{\Theta + 1} - 8}}, c_{\Pi,1} = \frac{-2\Theta^3 + 4\Theta^2 + 64\Theta + 64 - 4\sqrt{\Theta + 1} (\Theta^2 - 8\Theta - 16)}{\sqrt{(\Theta - 8)\Theta^3 ((\Theta - 4)\Theta - 8 (\sqrt{\Theta + 1} + 1))}},$$

and

$$c_{\Pi,2} = \frac{-2\Theta^3 + 12\Theta^2 - 32\Theta - 64 + 4 (\Theta^2 - 16) \sqrt{\Theta + 1}}{\sqrt{(\Theta - 8)\Theta^3 ((\Theta - 4)\Theta - 8 (\sqrt{\Theta + 1} + 1))}}.$$
REFERENCES


DEPARTMENT OF MATHEMATICAL SCIENCES, NEW JERSEY INSTITUTE OF TECHNOLOGY, NEWARK, NJ
Email address: aa2894@njit.edu

DEPARTMENT OF MATHEMATICAL SCIENCES, NEW JERSEY INSTITUTE OF TECHNOLOGY, NEWARK, NJ
Email address: goodman@njit.edu

DEPARTMENT OF MATHEMATICAL SCIENCES, NEW JERSEY INSTITUTE OF TECHNOLOGY, NEWARK, NJ
Email address: eo244@njit.edu