Complex Behavior in Coupled Nonlinear Waveguides

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Nonlinear Schrödinger/Gross-Pitaevskii Equation

\[ i\psi_t = -\nabla^2 \psi + V(r)\psi \pm |\psi|^2 \psi \]

Two contexts for today:

- **Propagation of light in a nonlinear waveguide**
  - \( \psi(x, z) \) gives the electric field envelope
  - “Evolution” occurs along axis of waveguide (\( t \rightarrow z \)) plus one transverse spatial dimension
  - Potential represents waveguide geometry

- **Evolution of a Bose-Einstein condensate (BEC)**
  - Everyone’s favorite nonlinear playground. A “new” state of matter achieved experimentally in the 1990’s.
  - One, two, or three space dimensions
  - Potential represents magnetic or optical trap
Periodic and chaotic tunneling in a 3-well waveguide

Why three wells?

- Other work on two-waveguide arrays shows symmetry-breaking bifurcations and an associated wobbling dynamics.
- Three waveguides provide the simplest system in which Hamiltonian Hopf bifurcations, which lead to complex dynamics, are possible.
- Significant interest in many-waveguide arrays. Useful to proceed: Simple Geometry → Complex Geometry, Simple Dynamics → Complex Dynamics
Stationary
\[ \psi(x, t) = \Psi(x)e^{-i\Omega t} \]
\[ \int_{\mathbb{R}} |\Psi(x)|^2 \, dx = \|\Psi\|_2^2 = N \]

Spontaneous symmetry breaking above critical intensity that is found analytically.
Kerr, Kevrekidis, Shlizerman, Weinstein 2008
see also Fukuzumi & Sacchetti 2011

Time-dependent dynamics
- Time dependent dynamics in a single or double well
- Rigorous result: long-time shadowing of ODE solutions by PDE solutions
  Marzuola & Weinstein 2010
  Pelinovsky & Phan 2012
  Goodman, Marzuola, Weinstein 2015

Experiment in Bose-Einstein condensate

Albiez et al. 2005

\[ V(x) = V_0(x + L) + V_0(x - L) \]
What got me thinking: Triple well

3-well potential & eigenfunctions

\[ V(x) = V_0(x + L) + V_0(x) + V_0(x - L) \]

Bifurcations of standing waves

(Kapitula/Kevrekidis/Chen SIADS 2006)

Mode unstable for range of \( N \)

Periodic Schrödinger Trimer


\[ \frac{d}{dt} \psi_n + C(\psi_{n-1} - 2\psi_n + \psi_{n+1}) + |\psi_n|^2 \psi_n = 0 \]

subject to \( \psi_{n+3} = \psi_n \)

“Hamiltonian Hopf Bifurcations”

Numerically-generated chaos
Two goals

- Understand what takes place at HH bifurcation as paradigm for nonlinear wave oscillatory instability.

- Flesh out the dynamics of relative periodic orbits in the system. *Eventual Goal:* Which of these dynamics can we prove exist?
Finite dimensional reduction

Decompose the solution as

\[ \psi = c_1(t)\Psi_1(t) + c_2(t)\Psi_2(t) + c_3(t)\Psi_3(t) + \eta(x, t) \]

projection onto eigenmodes \[ \eta(x, t) \perp \Psi_j(x) \]

Ignoring contribution of \( \eta(x, t) \) gives finite-dimensional Hamiltonian system with (approximate) Hamiltonian

\[ \bar{H} = \Omega_1 |c_1|^2 + \Omega_2 |c_2|^2 + \Omega_3 |c_3|^2 - A \left[ \frac{3}{2} \left( |c_1|^2 + |c_3|^2 \right)^2 + 2 |c_2|^4 + 4 |c_2|^2 |c_3 - c_1|^2 + \left( |c_1|^2 + |c_3|^2 \right) (c_1c_3 + \bar{c}_1\bar{c}_3) + \frac{3}{2} \left( c_1^2 \bar{c}_3^2 + \bar{c}_1^2 c_3^2 \right) + ((c_3 - c_1)^2 c_2^2 + (\bar{c}_3 - \bar{c}_1)^2 \bar{c}_2^2) \right] \]

For well-separated potential wells, the spectrum has the form

\( (\Omega_1, \Omega_2, \Omega_3) = (\Omega_2 - \Delta + \epsilon, \Omega_2, \Omega_2 + \Delta + \epsilon) \)

with \( \epsilon \ll \Delta \ll 1 \)
Symmetry reduction

System conserves squared $L^2$ norm $N$

- Reduces # of degrees of freedom from 3 to 2
- Removes fastest timescale

$$H_R = (-\Delta + \epsilon) |z_1|^2 + (\Delta + \epsilon) |z_3|^2 -$$
$$AN \left( z_1^2 + \bar{z}_1^2 + z_3^2 + \bar{z}_3^2 - 2(z_1 z_3 + \bar{z}_1 \bar{z}_3) - 4(z_1 \bar{z}_3 + \bar{z}_1 z_3) \right) -$$
$$A \left[ -\frac{1}{2} |z_1|^4 + 2 |z_1|^2 |z_3|^2 - \frac{1}{2} |z_3|^4 + \frac{3}{2} (z_1^2 \bar{z}_3^2 + \bar{z}_1^2 z_3^2) + \right.$$
$$\left( |z_1|^2 + |z_3|^2 \right) (5(z_1 \bar{z}_3 + \bar{z}_1 z_3) + 2(z_1 z_3 + \bar{z}_1 \bar{z}_3) - z_1^2 - \bar{z}_1^2 - z_3^2 - \bar{z}_3^2) \right]$$

- Relative fixed points in full system $\rightarrow$ fixed points in reduction
- Relative periodic orbits $\rightarrow$ periodic orbits

At $\epsilon = N = 0$, semisimple double frequency $i\Omega = \pm i\Delta$.

When $\epsilon > 0$, non-simple double eigenvalues at $N_{HH1} \approx \frac{\epsilon}{2A}$
and $N_{HH2} \approx \frac{\Delta - 2\epsilon}{2A}$, with instability in between.
Menagerie of standing waves

Three branches continue from linear system

Six branches arise in saddle-node bifurcations

Four stabilizations/destabilizations in HH bifurcations
Lyapunov Center Theorem: (Roughly) For each pair of imaginary eigenvalues of a fixed point, excepting resonance, there exists a one-parameter family of periodic orbits that limits to that fixed point.
Bifurcations in Hamiltonian systems change the topology of Lyapunov branches of periodic orbits

Standard Example: Hamiltonian Pitchfork \( \ddot{x} = \delta x + x^3 \)
ODE & PDE simulations

Trivial solution stable

Real($\lambda$)

Real($z_1$)

Poincaré Section

$|\psi(t)|$
ODE & PDE simulations

Chaotic heteroclinic bursting

Real($z_1$)  Poincaré Section  $|\psi(t)|$
ODE & PDE simulations

Real($\sigma_1$)

Poincaré Section

$|\psi(t)|$
Reduced Hamiltonian has 41 daunting terms!

\[ \tilde{H}_R = (-\Delta + \epsilon)|z_1|^2 + (\Delta + \epsilon)|z_3|^2 - AN \left( z_1^2 + \bar{z}_1^2 + z_3^2 + \bar{z}_3^2 - 2(z_1 z_3 + \bar{z}_1 \bar{z}_3) - 4(z_1 \bar{z}_3 + \bar{z}_1 z_3) \right) - \\
A \left[ -\frac{1}{2} |z_1|^4 + 2 |z_1|^2 |z_3|^2 - \frac{1}{2} |z_3|^4 + \frac{3}{2} (z_1^2 \bar{z}_3^2 + \bar{z}_1^2 z_3^2) + \\
\left(|z_1|^2 + |z_3|^2\right) \left(5(z_1 \bar{z}_3 + \bar{z}_1 z_3) + 2(z_1 z_3 + \bar{z}_1 \bar{z}_3) - z_1^2 - \bar{z}_1^2 - z_3^2 - \bar{z}_3^2 \right) \right]. \]

Goal: understand periodic orbits of \( \tilde{H}_R \) using Hamiltonian Normal Forms

Given a system with Hamiltonian \( H = H_0(z) + \epsilon \tilde{H}(z, \epsilon) \)
find a near-identity canonical transformation \( z = \mathcal{F}(y, \epsilon) \)
such that the transformed Hamiltonian
\[ K(y, \epsilon) = H(\mathcal{F}(y, \epsilon), \epsilon) = H_0(y) + \epsilon \tilde{K}(y, \epsilon) \]
is “simpler” than \( H(z, \epsilon) \).
What does “simpler” mean?

- Try to remove terms from $H$ to construct $K$
- Eliminating terms at a given order in $\varepsilon, y$ introduces new terms of higher order
- A term can be removed if it lies in the range of the adjoint operator of $\text{ad}_{H_0} = \{ \cdot, H_0 \}$.
- Invoke Fredholm alternative. Resonant terms in adjoint null space. Project Hamiltonian onto this subspace.
- For example in our problem

\[
\text{ad}_{H_0} = \{ \cdot, H_0 \} = \{ \cdot, H_0 \}
\]

\[
\alpha_1 \quad \alpha_3 \\
0 \quad 0 \quad \bar{z}_1 \bar{z}_3 \quad |z_3|^2 \\
1 \quad \bar{z}_1 \bar{z}_3 \quad |z_1|^2 z_1 z_3
\]

(a) Degree Two

\[
\begin{array}{c|c|c|c}
\alpha_1 & \alpha_3 & 0 & 1 \\
0 & \bar{z}_1 \bar{z}_3 & |z_3|^2 & 1 \\
1 & \bar{z}_1 \bar{z}_3 & |z_1|^2 z_1 z_3 & 2 \\
2 & \bar{z}_1 \bar{z}_3 & |z_1|^4 & 1 \\
\end{array}
\]

(b) Degree Four
Three normal form calculations

• Semisimple -1:1 resonance for $\epsilon \ll 1$, $N = O(\epsilon)$
  Gives HH1 at $N_{\text{crit}} = \frac{\epsilon}{2A} + O(\epsilon^2)$

• Nonsemisimple -1:1 resonance at $N_{\text{crit}}$ using a further simplification of above normal form

• Nonsemisimple -1:1 resonance computed numerically at numerical location of HH2
Normal form near semisimple double eigenvalue (Chow/Kim 1988)

\[ H = -\Delta |z_1|^2 + \Delta |z_3|^2 \]

Normal Form

\[ H_{\text{norm}} = -\Delta |z_1|^2 + \Delta |z_3|^2 + \epsilon \left( |z_1|^2 + |z_3|^2 \right) + 2AN(z_1z_3 + \bar{z}_1\bar{z}_3) + A \left[ \frac{1}{2}|z_1|^4 - 2|z_1|^2|z_3|^2 + \frac{1}{2}|z_3|^4 - 2 \left( |z_1|^2 + |z_3|^2 \right) (z_1z_3 + \bar{z}_1\bar{z}_3) \right] \]

In Canonical Polar Coordinates

\[ H = \Delta (-J_1 + J_3) + \epsilon (J_1 + J_3) + 4AN \sqrt{J_1J_3} \cos (\theta_1 + \theta_3) + A \left( \frac{1}{2}J_1^2 - 2J_1J_3 + \frac{1}{2}J_3^2 - 4\sqrt{J_1J_3}(J_1 + J_3) \cos (\theta_1 + \theta_3) \right) \]

Independent of \((\theta_1 - \theta_3)\) implying the existence of a conserved quantity and the integrability of the Normal Form.

Advantage: Easier to find solution structure in Normal Form.
The system can be further reduced. Periodic orbits \( \left( \begin{array}{c} J_1 \\ J_3 \end{array} \right) e^{i\Omega t} \) solve:

\[
\sqrt{J_1 J_3} \left( 2\epsilon - A (J_1 + J_3) \right) + 2A \left( N (J_1 + J_3) - J_1^2 - 6J_1 J_3 - J_3^2 \right) \cos \Theta = 0 \\
\sqrt{J_1 J_3} \left( N - J_1 - J_3 \right) \sin \Theta = 0
\]

With \( \Theta = (\theta_1 + \theta_3) \)

\( J_1 \) and \( J_3 \) act as barycentric coordinates on the triangle of admissible solutions showing relative strength of the three modes.
Sequence of bifurcations in Normal Form

$0 < N < \frac{2 \epsilon}{5A}$

2 Lyapunov families of fixed points + unphysical branch

$\frac{2 \epsilon}{5A} < N < \frac{\epsilon}{2A}$

Unphysical branches cross into physical region

$\frac{\epsilon}{2A} < N < \frac{2 \epsilon}{A}$

Lyapunov branches “pinch off”

Question: At second bifurcation point HH2, must have Lyapunov families of fixed point. Where do they come from?
Normal form for non-semisimple 1:1 resonances at HH1 and HH2 (Meyer-Schmidt 1974)

In symplectic polar coordinates \((r, \theta, p_r, p_\theta)\), this is:

\[
H = H_0(r, p_r, p_\theta) + \mu^2 \delta H_2(r, p_\theta) + H_4(r, p_\theta)
\]

\[
= \Omega p_\theta + \frac{\sigma}{2} \left( p_r^2 + \frac{p_\theta^2}{r^2} \right) + \mu^2 \delta \left( \alpha p_\theta + \frac{b}{2} r^2 \right) + \frac{c}{2} p_\theta^2 + \frac{d}{2} p_\theta r^2 + \frac{e}{8} r^4
\]

\(\delta = \pm 1, \mu \ll 1\)

Poincaré-Lindstedt argument: periodic orbits with “amplitude” \(\mu r\) and frequency \(\Omega + \mu \omega_1\) when there is a solution to

\[
2\omega_1^2 - \sigma er^2 = 2\delta \sigma \beta
\]

Two cases:

Hyperbolic \(\sigma e > 0\)

Elliptic \(\sigma e < 0\)
The bifurcation at HH1

Computations using previous normal form

Increasing $N \Rightarrow$

Numerically Computed Periodic orbits (not normal form)
Some computed PDE solutions on this branch

\[(x_1(0)^2 + x_3(0)^2)/2\]
The bifurcation at HH2

Numerically Computed Periodic orbits

New family of periodic orbits arises in “elliptic” HH bifurcation
Increasing $N$

Solutions must satisfy $|z_1|^2 + |z_3|^2 < N$.

PDE

$N = 0.82$

ODE

$N = 0.8135$
What's going on?

Getting close to other fixed points

What about the other Lyapunov branches of periodic orbits?

\( N = 0.8125 \)
I thought saddle-node bifurcations were boring
Normal form for $0^{2i}\omega$ bifurcation

$$H = \left(\frac{q_1^2}{2} + \frac{p_1^2}{2}\right) + \alpha \left(\frac{p_2^2}{2} + \delta q_2 - \frac{q_2^3}{3}\right) + \beta Iq_2 + H_{\text{higher}}(q_2, I) + R_\infty(q_1, q_2, p_1, p_2)$$

where $I = \left(\frac{q_1^2}{2} + \frac{p_1^2}{2}\right)$

Three families of periodic orbits:
- Fast
- Slow
- Mixed

Fast & Oscillatory
Saddle node bifurcation at $\delta = 0$
Coupling
Small beyond all orders remainder
Perturbation expansion shows two regimes:

- \( N < N_{\text{crit}} \)
- \( N > N_{\text{crit}} \)

Saddle-node 1

Saddle-node 2

Mixed periodic orbits bifurcate when \( \delta = \frac{1}{\alpha^4 n^4} \)
Saddle-node I

Gelfreich-Lerman 2003
Saddle-node 2

(a) $N=0.664$

(b) $N=0.67$

(c) $N=0.673$
Parting Words

• This problem has an ODE part and a PDE part
• Increasing from two wells to three makes the ODE part of the problem hard
• In addition to standing waves, there is a whole lot of additional structure in solutions that oscillate among the three waveguides
• Normal forms give us a partial picture of the reduced dynamics
• Even saddle-node bifurcations are interesting.
• Big question: What can be proven about shadowing these orbits in NLS/GP?

For re/preprints http://web.njit.edu/~goodman
Thanks!