Complex Behavior in Coupled Nonlinear Waveguides

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NJIT

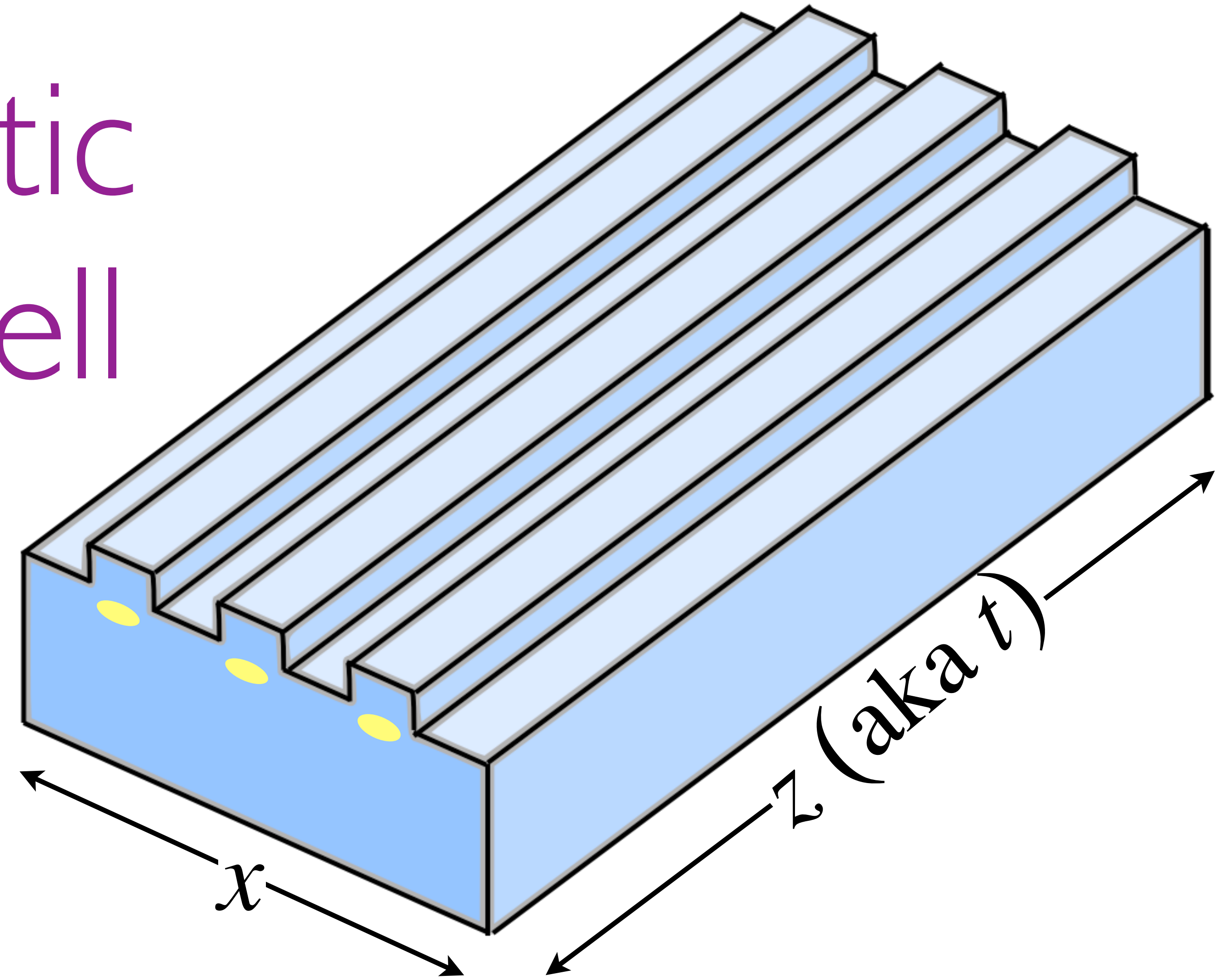
Nonlinear Schrödinger/Gross-Pitaevskii Equation

$$i\psi_t = -\nabla^2\psi + V(r)\psi \pm |\psi|^2\,\psi$$

Two contexts for today:

- Propagation of light in a nonlinear waveguide
  - $\psi(x,z)$ gives the electric field envelope
  - "Evolution" occurs along axis of waveguide ($t \to z$) plus one transverse spatial dimension
  - Potential represents waveguide geometry
- Evolution of a Bose-Einstein condensate (BEC)
  - Everyone's favorite nonlinear playground. A "new" state of matter achieved experimentally in the 1990's.
  - One, two, or three space dimensions
  - Potential represents magnetic or optical trap

Periodic and chaotic tunneling in a 3-well waveguide



Why three wells?

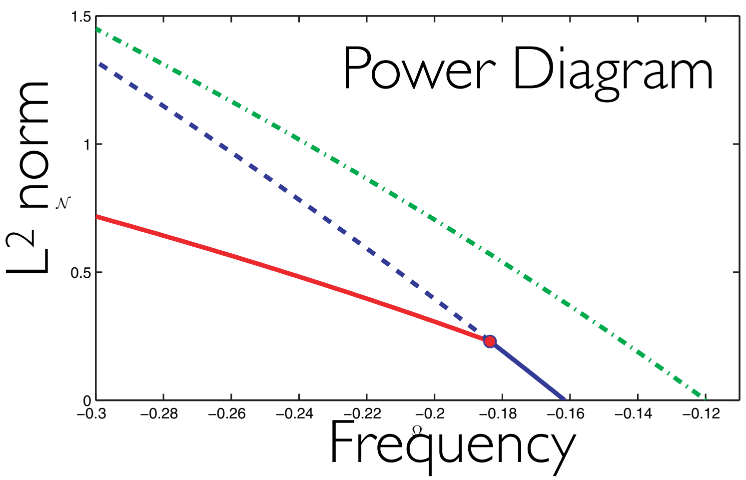
- Other work on two-waveguide arrays shows symmetry-breaking bifurcations and an associated wobbling dynamics.
- Three waveguides provide the simplest system in which Hamiltonian Hopf bifurcations, which lead to complex dynamics, are possible.
- Significant interest in many-waveguide arrays. Useful to proceed: Simple Geometry → Complex Geometry, Simple Dynamics → Complex Dynamics

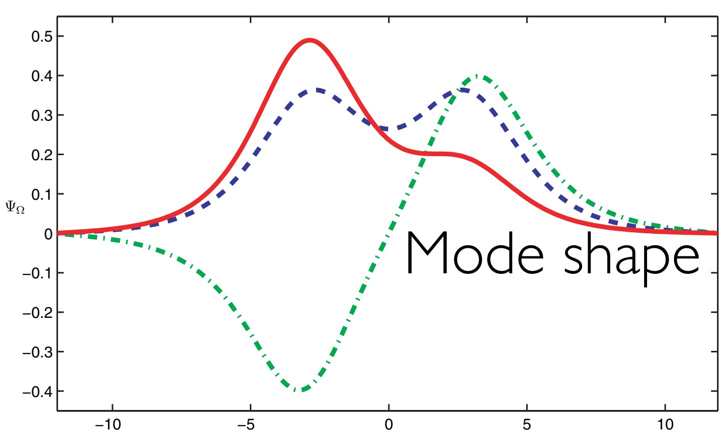
What got me thinking: Double well $V(x) = V_0(x+L) + V_0(x-L)$

Stationary

$$\psi(x,t) = \Psi(x)e^{-i\Omega t}$$

$$\int_{\mathbb{R}} \Psi(x)^2 dx = \|\Psi\|_2^2 = \mathcal{N}$$



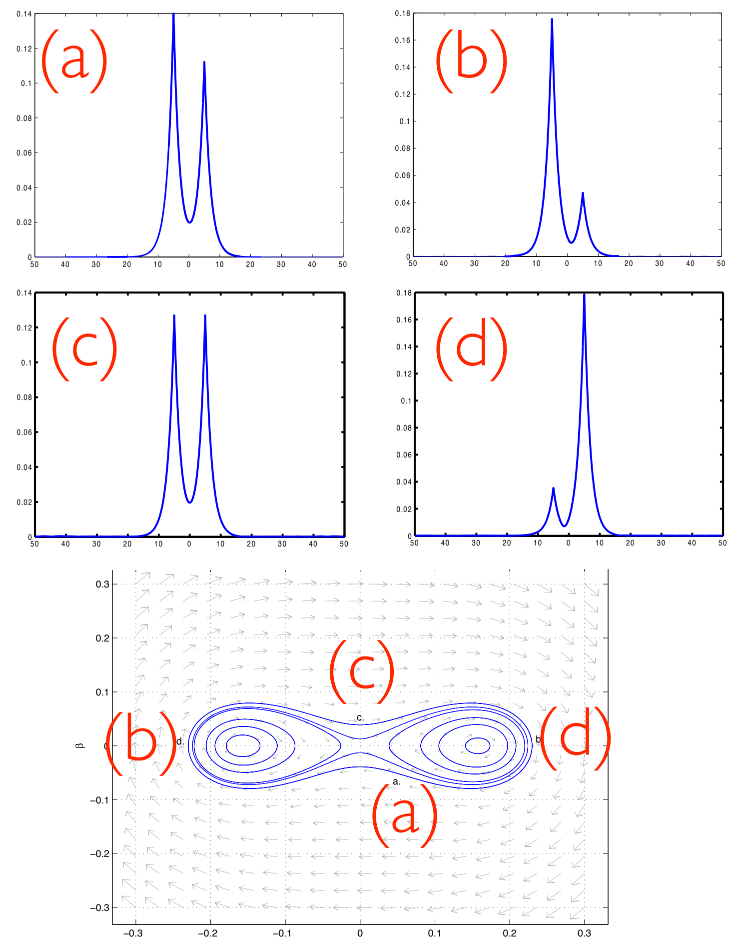


Spontaneous symmetry breaking above critical intensity that is found analytically.

Kirr, Kevrekidis, Shlizerman, Weinstein 2008

see also Fukuizumi & Sacchetti 2011

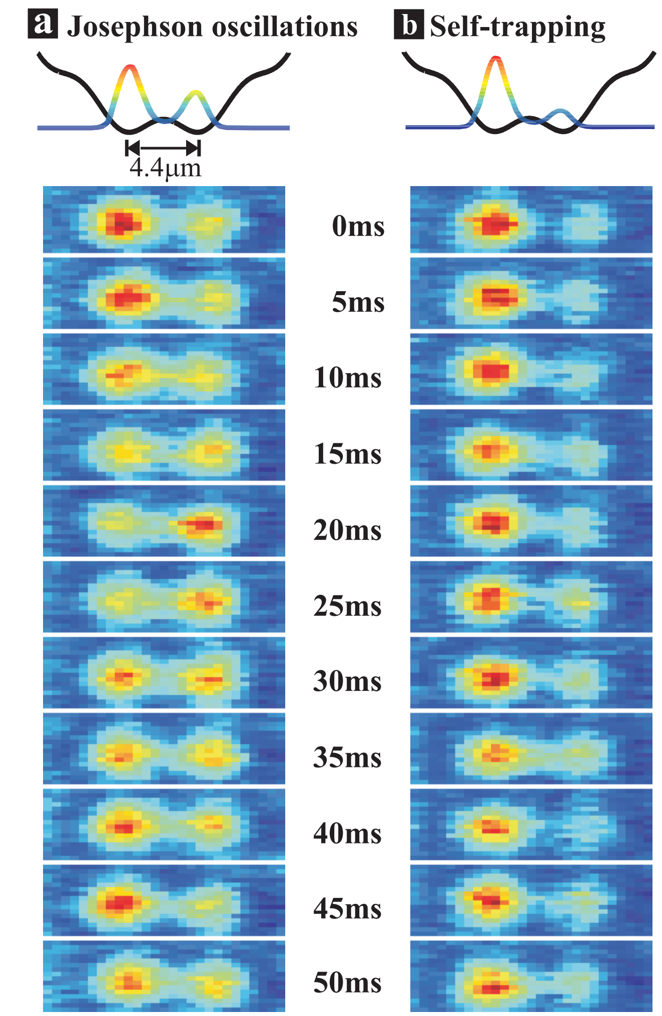
Time-dependent dynamics



- Time dependent dynamics in a single or double well
- Rigorous result: long-time shadowing of ODE solutions by PDE solutions

Marzuola & Weinstein 2010
Pelinovsky & Phan 2012
Goodman, Marzuola, Weinstein 2015

Experiment in Bose-Einstein condensate

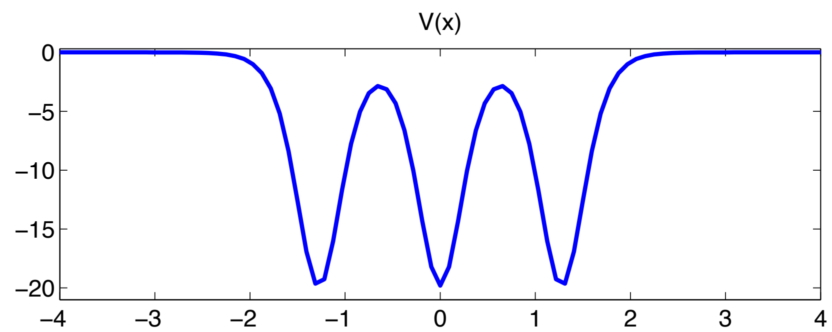


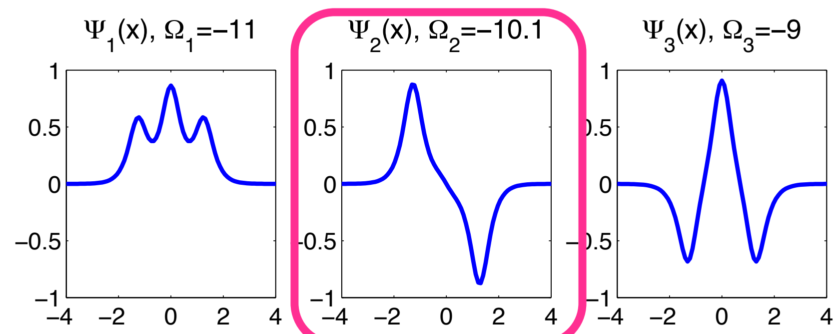
Albiez et al. 2005

What got me thinking: Triple well

3-well potential & eigenfunctions

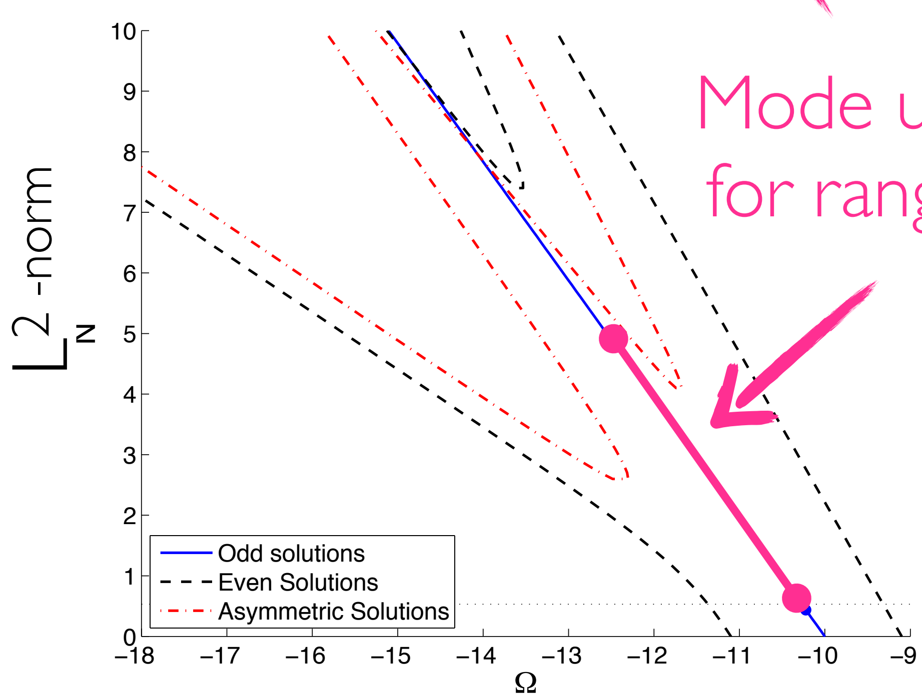
$$V(x) = V_0(x+L) + V_0(x) + V_0(x-L)$$





Bifurcations of standing waves

(Kapitula/Kevrekidis/Chen SIADS 2006)



Mode unstable for range of *N*

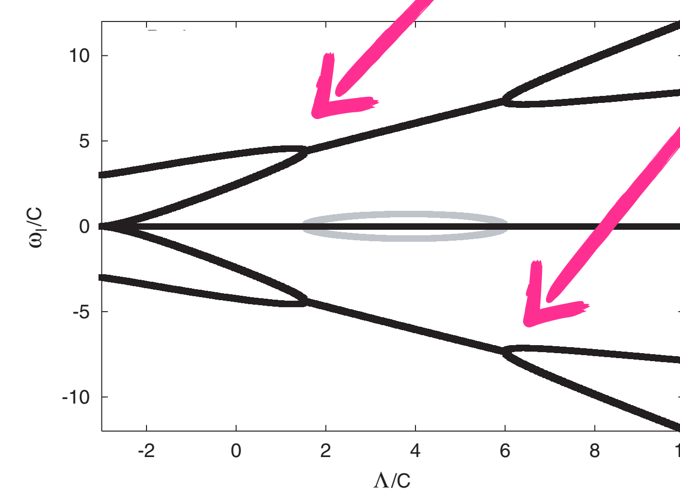
Periodic Schrödinger Trimer

(Johansson J. Phys. A 2004)

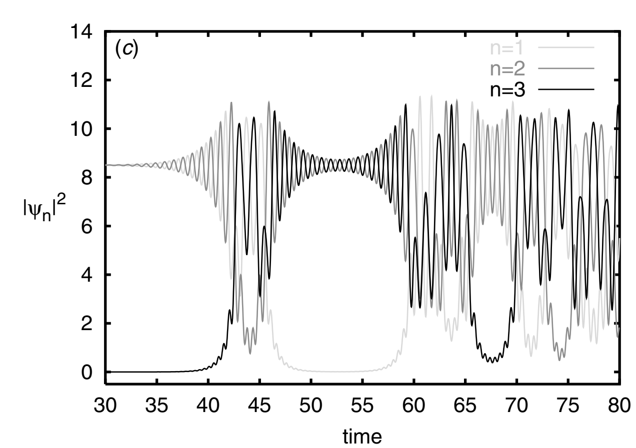
$$\frac{d}{dt}\psi_n + C(\psi_{n-1} - 2\psi_n + \psi_{n+1}) + |\psi_n|^2\psi_n = 0$$

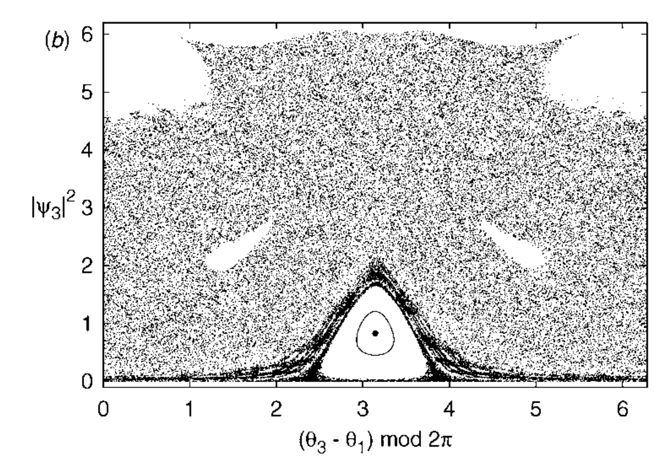
$$\text{subject to } \psi_{n+3} = \psi_n$$

"Hamiltonian Hopf Bifurcations"



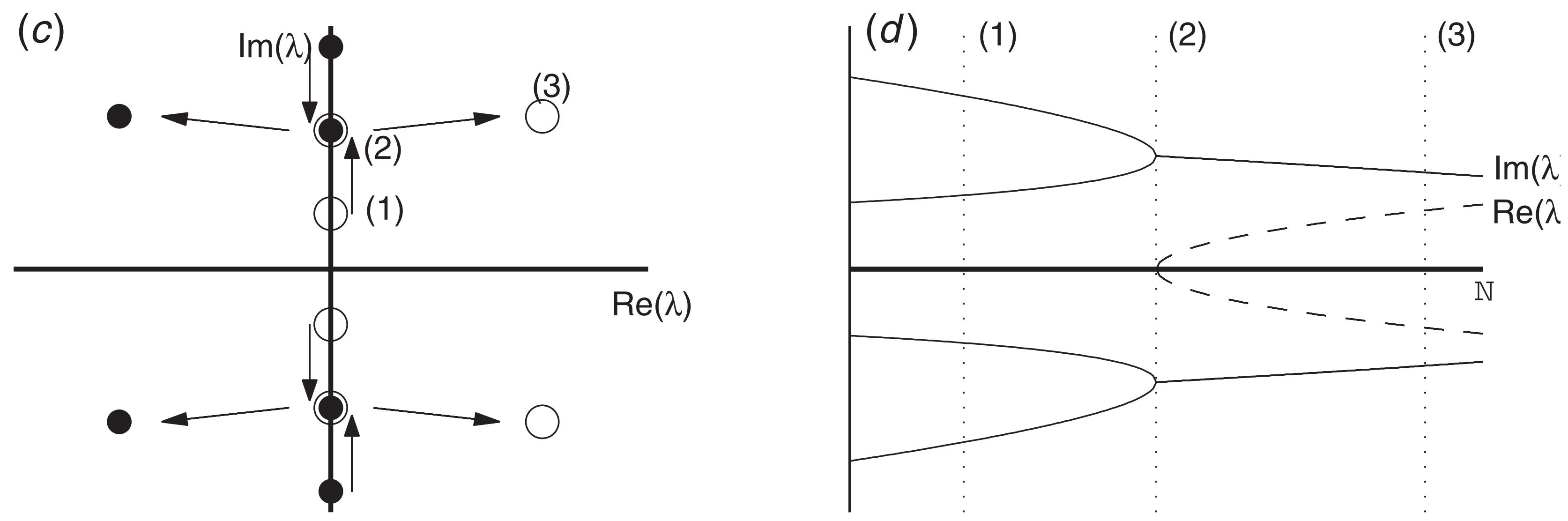
Numerically-generated chaos





Two goals

- Understand what takes place at HH bifurcation as paradigm for nonlinear wave oscillatory instability.



- Flesh out the dynamics of relative periodic orbits in the system. *Eventual Goal:* Which of these dynamics can we prove exist?

Finite dimensional reduction

Decompose the solution as

$$\psi = \boxed{c_1(t)\Psi_1(t) + c_2(t)\Psi_2(t) + c_3(t)\Psi_3(t)} + \boxed{\eta(x,t)}$$

projection onto eigenmodes

$\eta(x,t) \perp \Psi_j(x)$

Ignoring contribution of $\eta(x,t)$ gives finite-dimensional Hamiltonian system with (approximate) Hamiltonian

$$\bar{H} = \Omega_1 |c_1|^2 + \Omega_2 |c_2|^2 + \Omega_3 |c_3|^2 - A\Big[\tfrac{3}{2}\left(|c_1|^2 + |c_3|^2\right)^2 + 2|c_2|^4 + 4|c_2|^2 |c_3 - c_1|^2 + \left(|c_1|^2 + |c_3|^2\right)(c_1 c_3 + \bar{c}_1 \bar{c}_3) + \tfrac{3}{2}\left(c_1^2 \bar{c}_3^2 + \bar{c}_1^2 c_3^2\right) + \left((c_3 - c_1)^2 \bar{c}_2^2 + (\bar{c}_3 - \bar{c}_1)^2 c_2^2\right)\Big]$$

For well-separated potential wells, the spectrum has the form

$$(\Omega_1, \Omega_2, \Omega_3) = (\Omega_2 - \Delta + \epsilon, \Omega_2, \Omega_2 + \Delta + \epsilon)$$

with $\epsilon \ll \Delta \ll 1$

Symmetry reduction

System conserves squared L² norm N

- Reduces # of degrees of freedom from 3 to 2
- Removes fastest timescale

$$
\begin{aligned}
\bar{H}_{\mathrm{R}} =& (-\Delta+\epsilon)\left|z_1\right|^2+(\Delta+\epsilon)\left|z_3\right|^2- \\
& AN\left(z_1^2+\bar{z}_1^2+z_3^2+\bar{z}_3^2-2(z_1 z_3+\bar{z}_1 \bar{z}_3)-4(z_1 \bar{z}_3+\bar{z}_1 z_3)\right)- \\
& A\Big[-\tfrac{1}{2}\left|z_1\right|^4+2\left|z_1\right|^2\left|z_3\right|^2-\tfrac{1}{2}\left|z_3\right|^4+\tfrac{3}{2}(z_1^2 \bar{z}_3^2+\bar{z}_1^2 z_3^2)+ \\
& \left(\left|z_1\right|^2+\left|z_3\right|^2\right)\left(5(z_1 \bar{z}_3+\bar{z}_1 z_3)+2(z_1 z_3+\bar{z}_1 \bar{z}_3)-z_1^2-\bar{z}_1^2-z_3^2-\bar{z}_3^2\right).\Big]
\end{aligned}
$$

- Relative fixed points in full system $\longrightarrow$ fixed points in reduction
- Relative periodic orbits $\longrightarrow$ periodic orbits

At $\epsilon = N = 0$, semisimple double frequency $i\Omega = \pm i\Delta$.

When $\epsilon > 0$, non-simple double eigenvalues at $N_{\mathrm{HH1}} \approx \dfrac{\epsilon}{2A}$

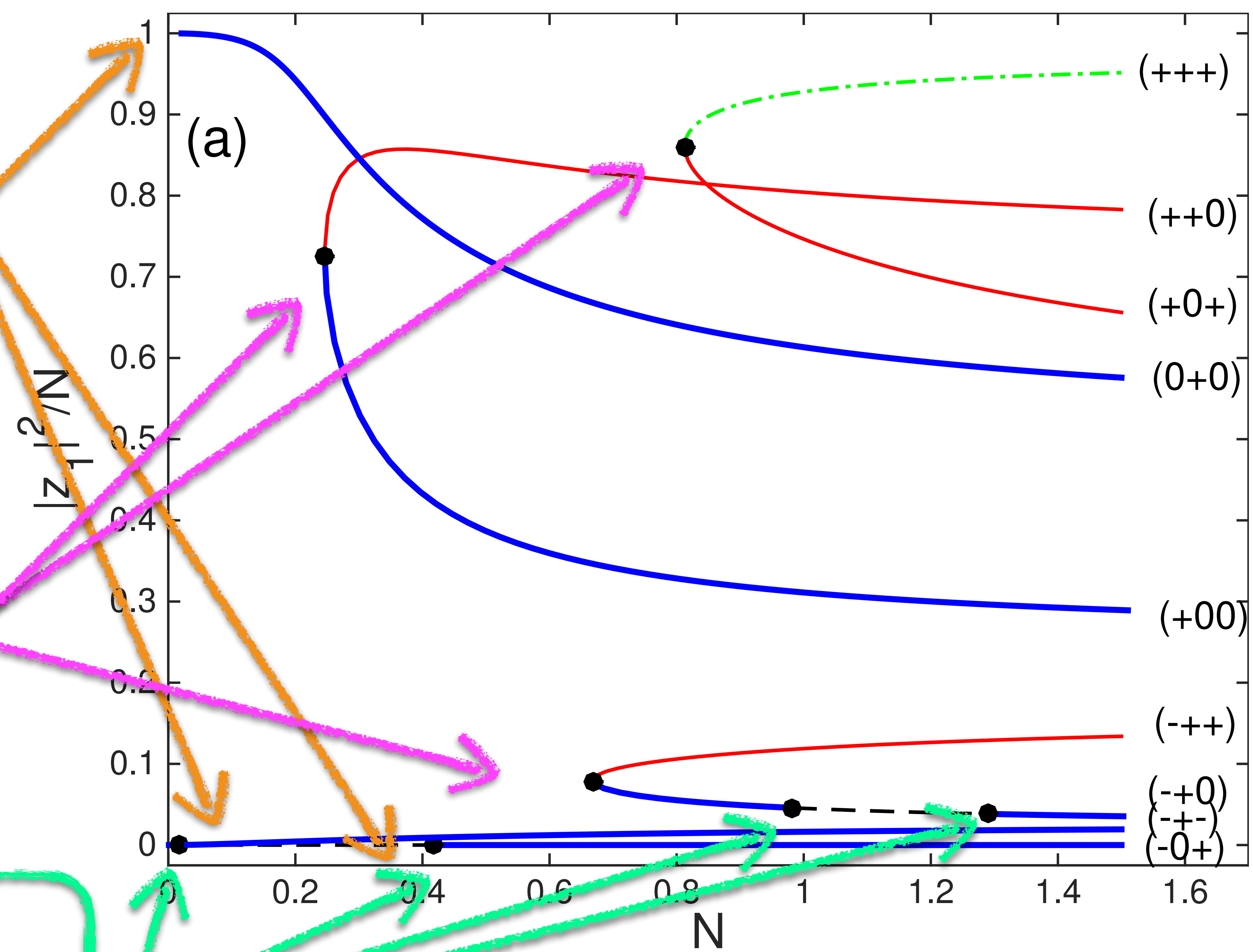
and $N_{\mathrm{HH2}} \approx \dfrac{\Delta - 2\epsilon}{2A}$, with instability in between.

Menagerie of standing waves

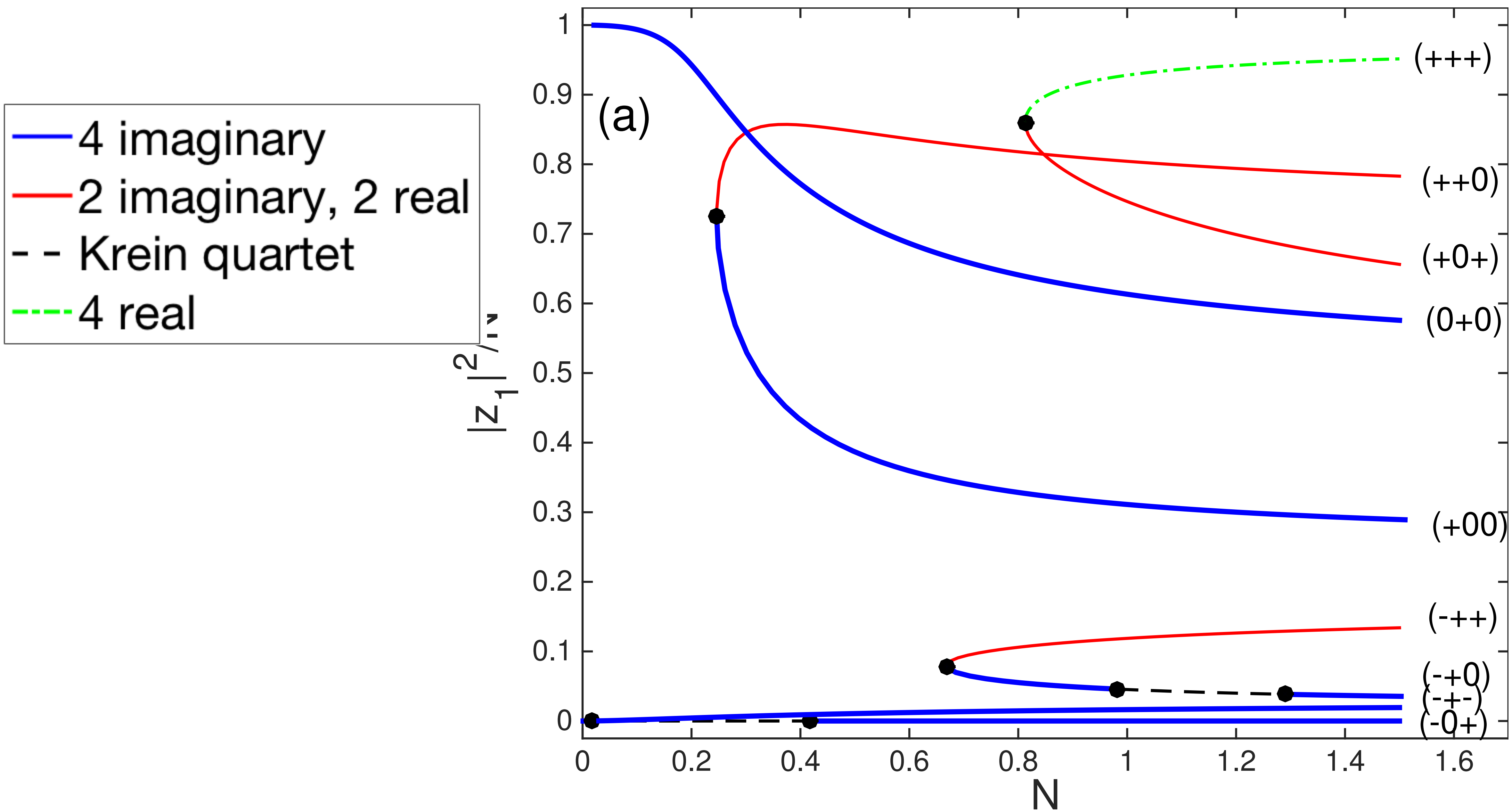
Three branches continue from linear system

Six branches arise in saddle-node bifurcations

Four stabilizations/ destabilizations in HH bifurcations



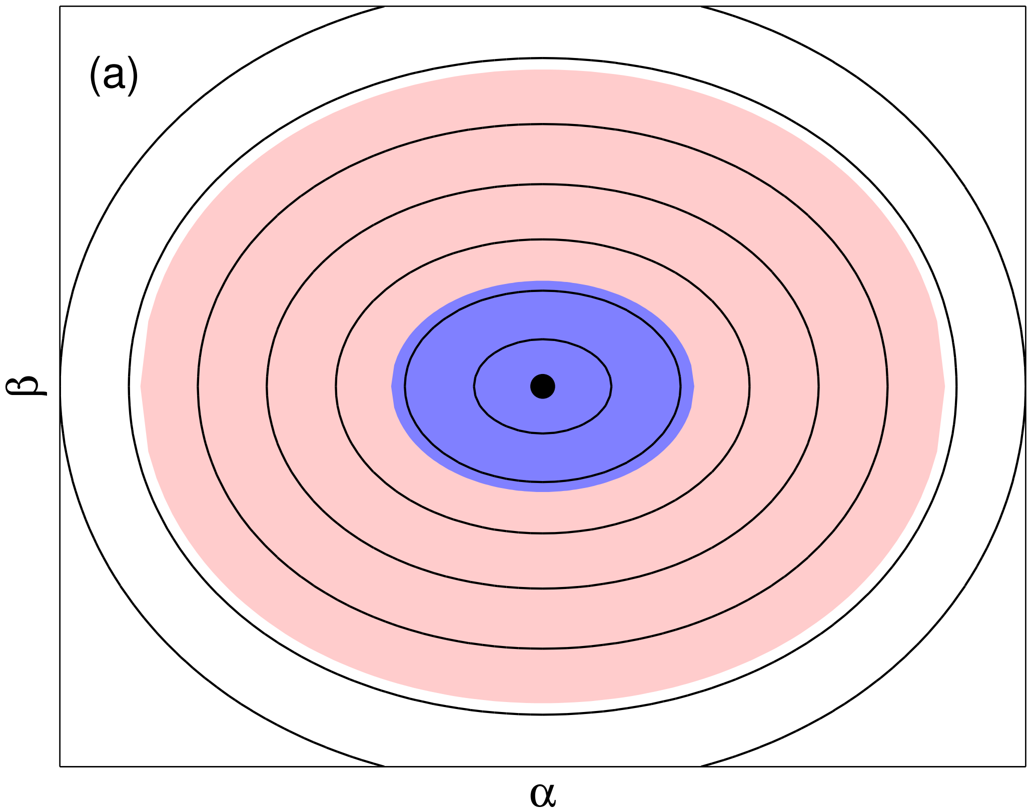
More about this picture



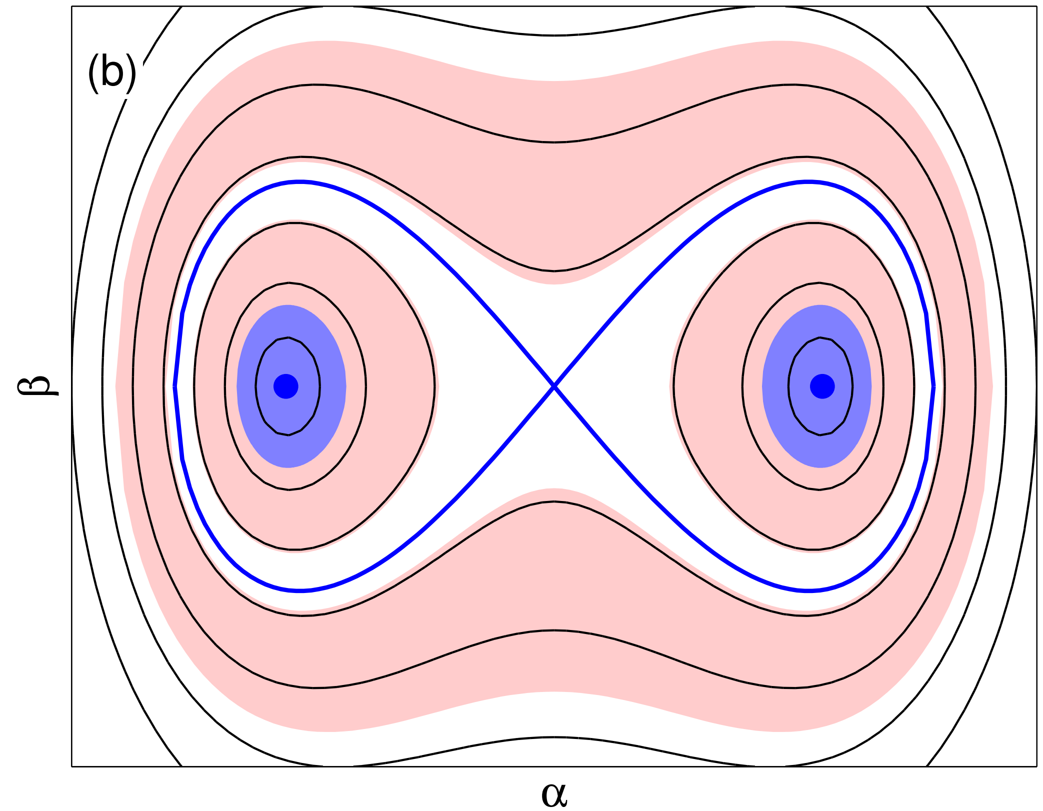
Lyapunov Center Theorem: (Roughly) For each pair of imaginary eigenvalues of a fixed point, excepting resonance, there exists a one-parameter family of periodic orbits that limits to that fixed point.

Bifurcations in Hamiltonian systems change the topology of Lyapunov branches of periodic orbits

Standard Example: Hamiltonian Pitchfork $\ddot{x} = \delta x + x^3$



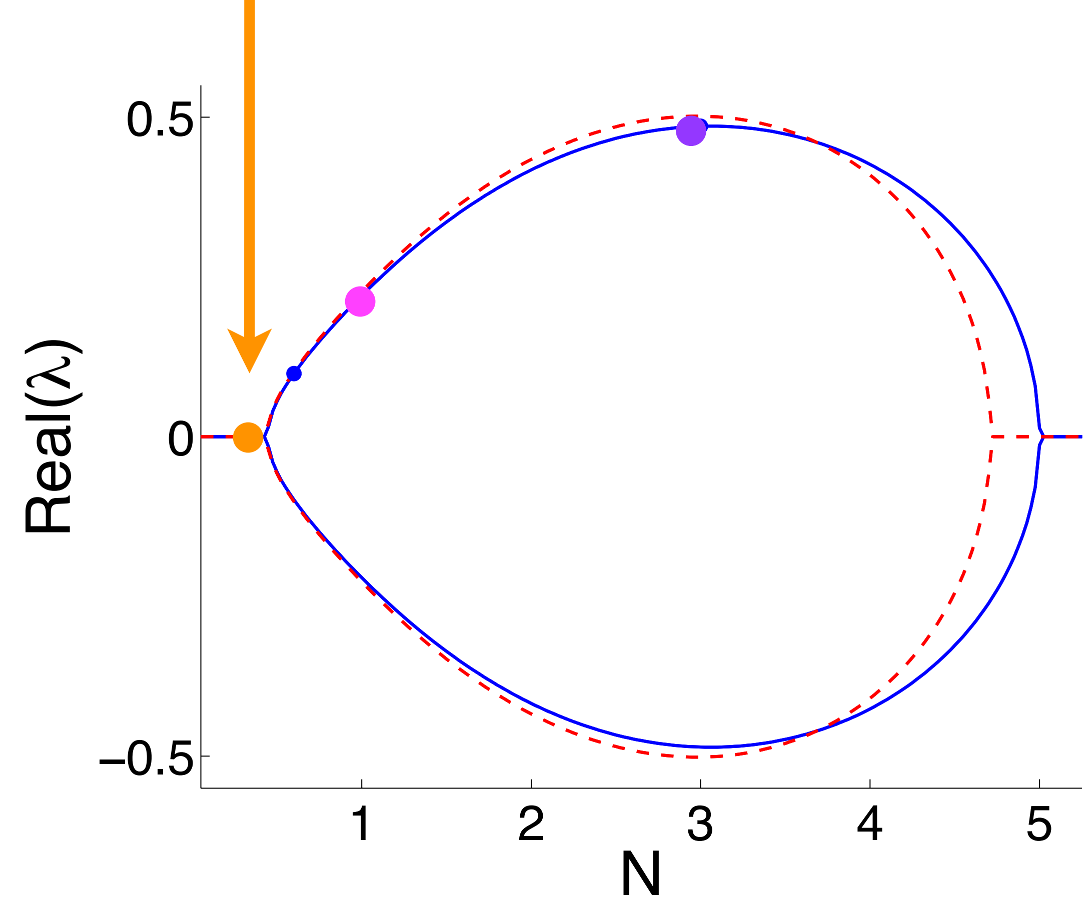
$\delta > 0$

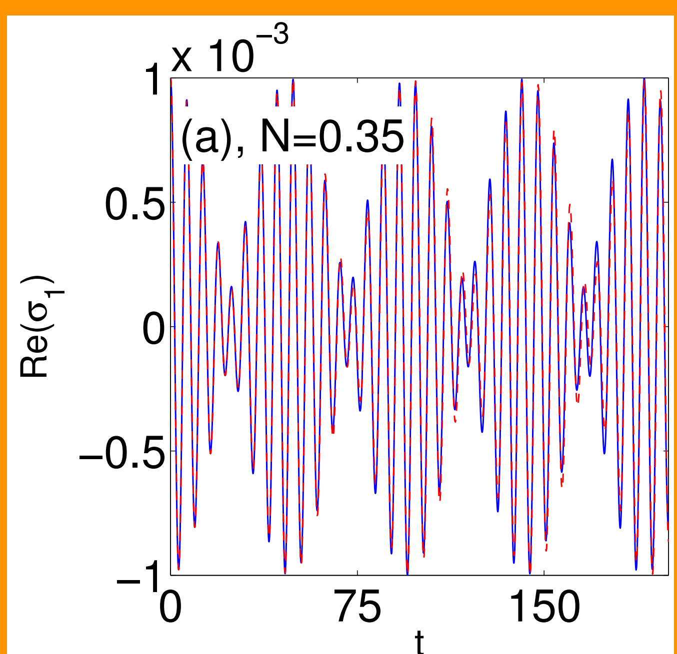


$\delta < 0$

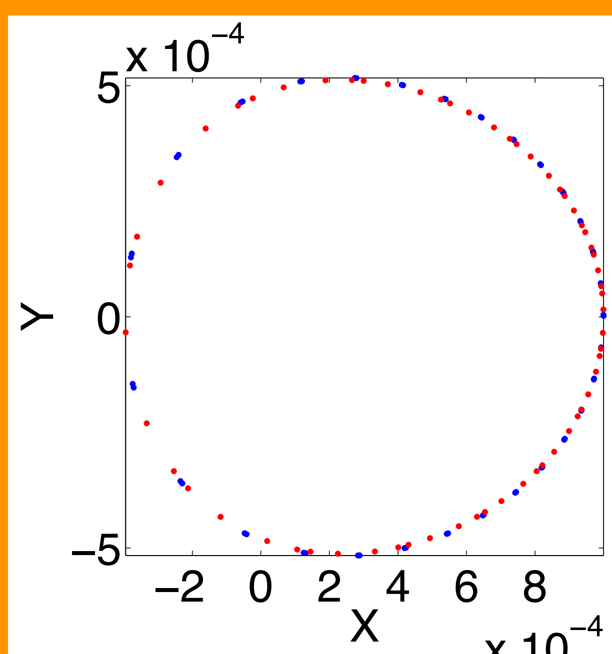
ODE & PDE simulations

Trivial solution stable

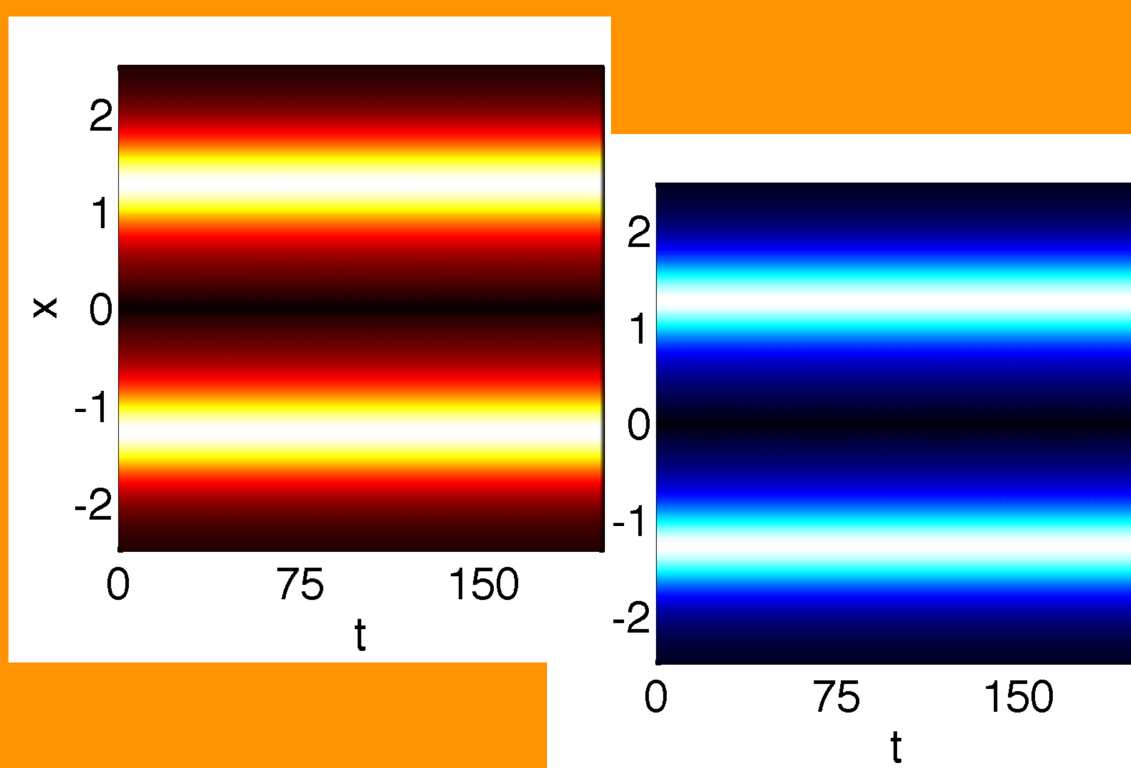




Real(z_1)



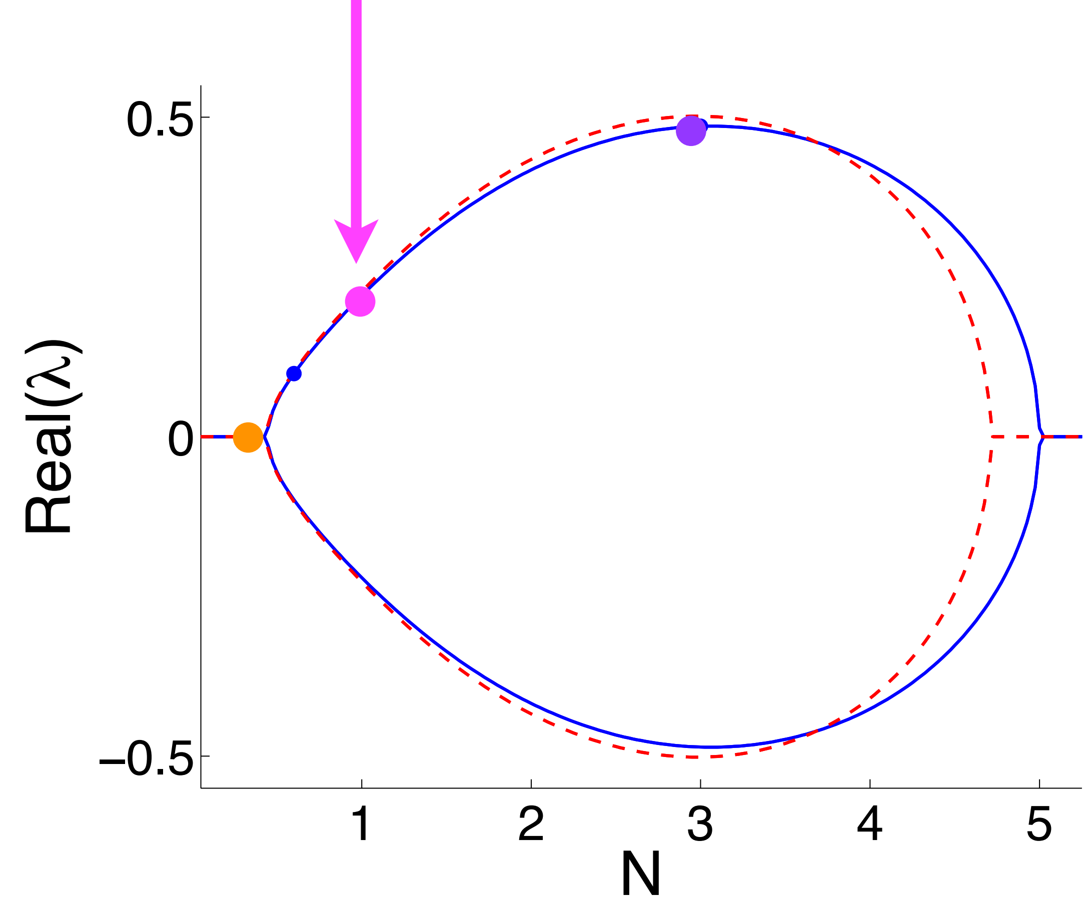
Poincaré Section

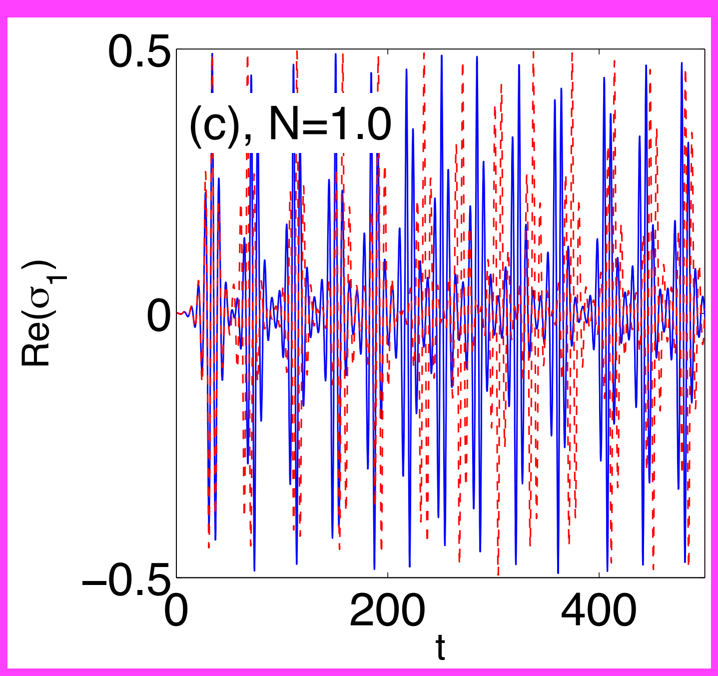


$|\psi(t)|$

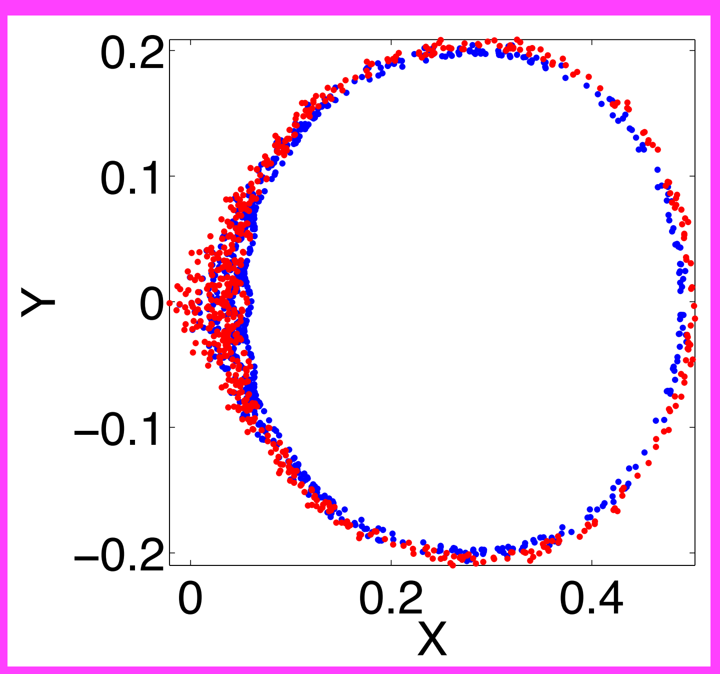
ODE & PDE simulations

Chaotic heteroclinic bursting

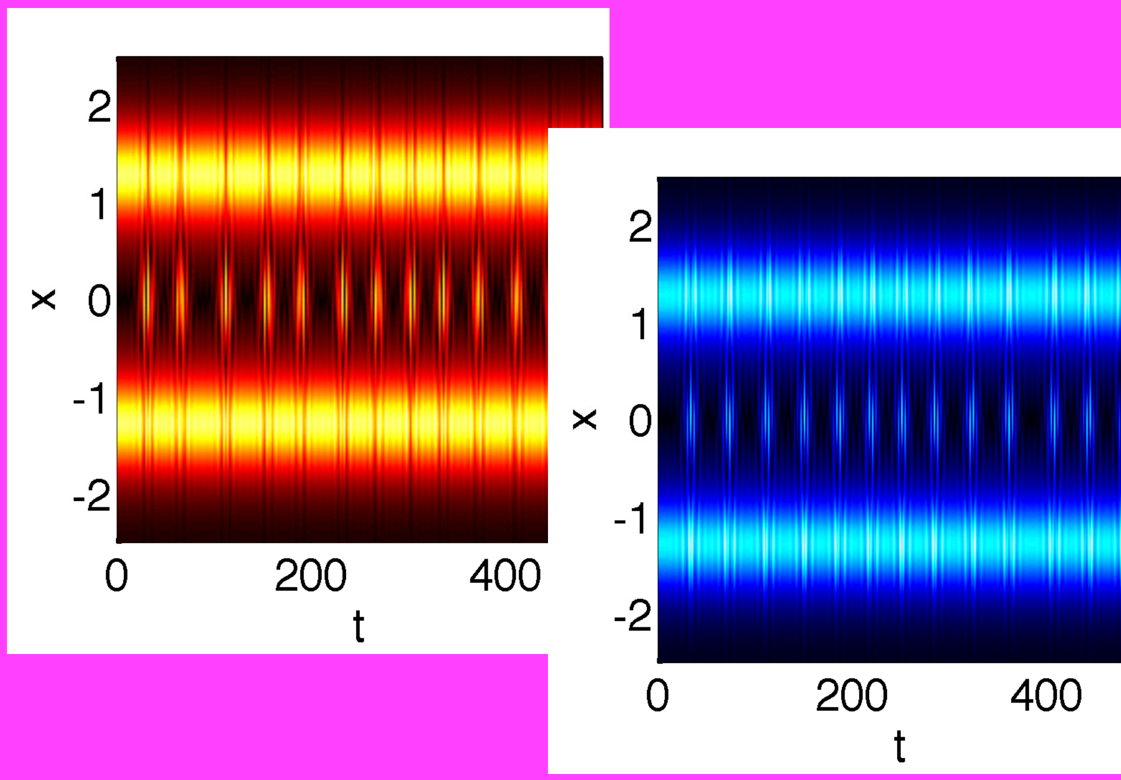




Real(z_1)

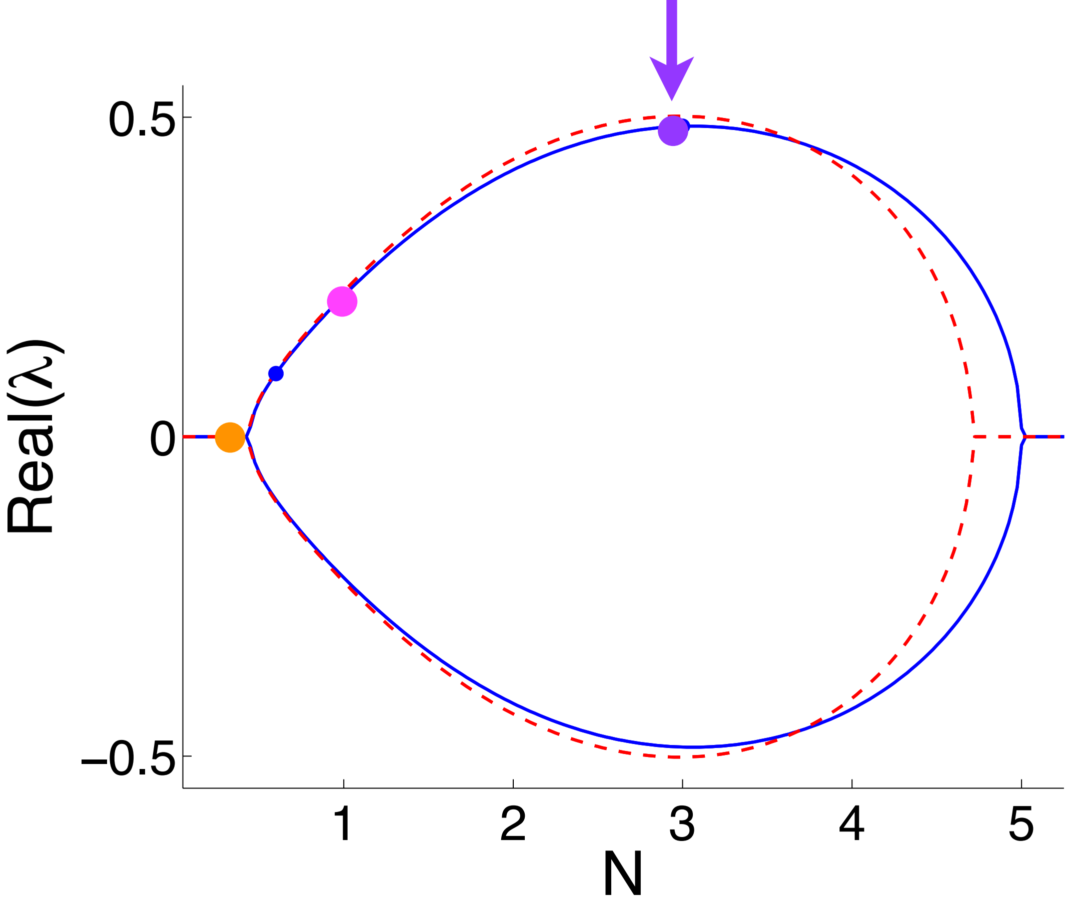


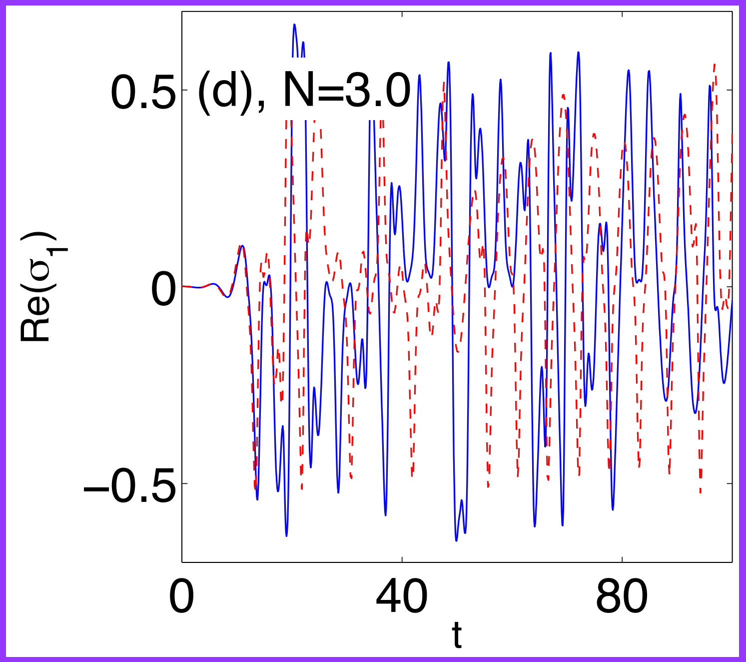
Poincaré Section



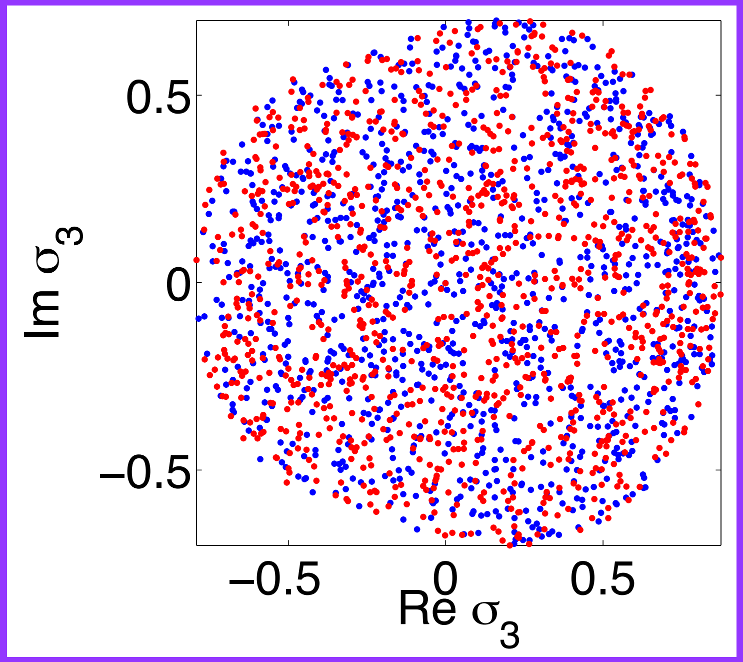
$|\psi(t)|$

ODE & PDE simulations

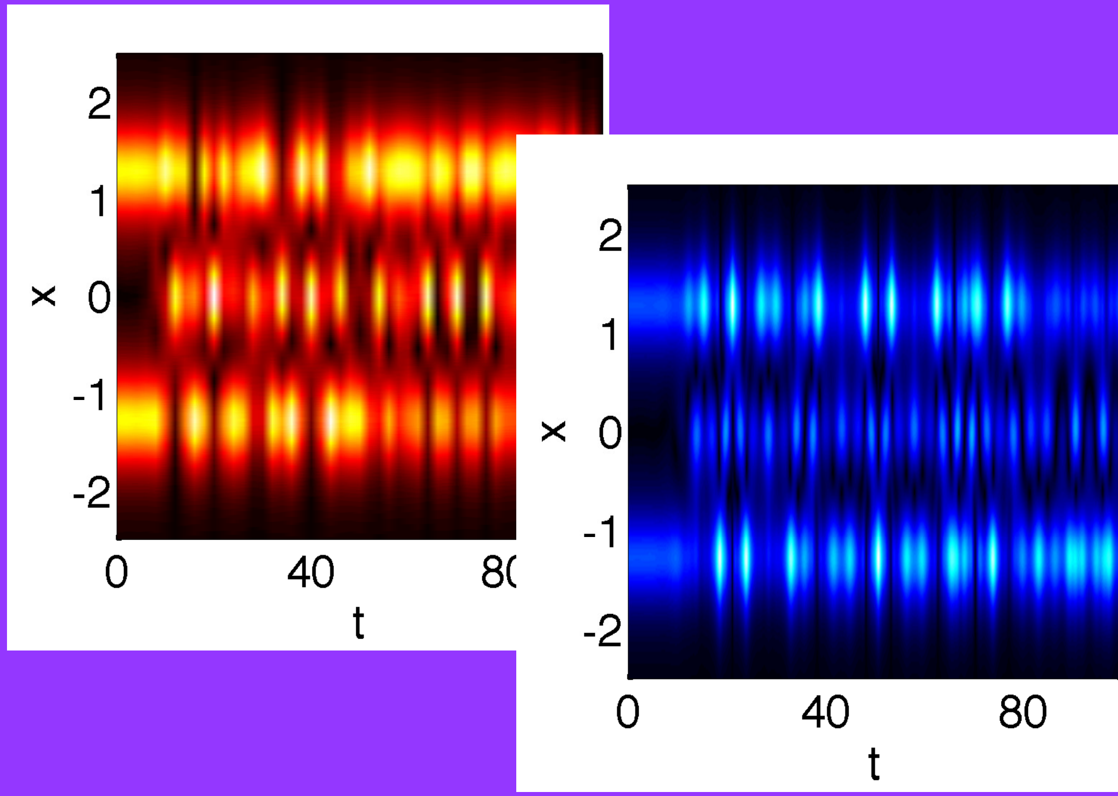




Real(z_1)



Poincaré Section



$|\psi(t)|$

Reduced Hamiltonian has 41 daunting terms!

$$\begin{aligned}\bar{H}_{\mathrm{R}} =&(-\Delta+\epsilon)\left|z_1\right|^2+(\Delta+\epsilon)\left|z_3\right|^2- \\ &AN\left(z_1^2+\bar{z}_1^2+z_3^2+\bar{z}_3^2-2(z_1z_3+\bar{z}_1\bar{z}_3)-4(z_1\bar{z}_3+\bar{z}_1z_3)\right)- \\ &A\Big[-\tfrac{1}{2}\left|z_1\right|^4+2\left|z_1\right|^2\left|z_3\right|^2-\tfrac{1}{2}\left|z_3\right|^4+\tfrac{3}{2}(z_1^2\bar{z}_3^2+\bar{z}_1^2z_3^2)+ \\ &\left(\left|z_1\right|^2+\left|z_3\right|^2\right)\left(5(z_1\bar{z}_3+\bar{z}_1z_3)+2(z_1z_3+\bar{z}_1\bar{z}_3)-z_1^2-\bar{z}_1^2-z_3^2-\bar{z}_3^2\right).\Big]\end{aligned}$$

Goal: understand periodic orbits of $\bar{H}_{\mathrm{R}}$ using Hamiltonian Normal Forms

Given a system with Hamiltonian $H=H_0(z)+\epsilon\tilde{H}(z,\epsilon)$

find a near-identity canonical transformation $z=\mathcal{F}(y,\epsilon)$

such that the transformed Hamiltonian

$$K(y,\epsilon)=H\left(\mathcal{F}(y,\epsilon),\epsilon\right)=H_0(y)+\epsilon\tilde{K}(y,\epsilon)$$

is "simpler" than $H(z,\epsilon)$.

What does "simpler" mean?

- Try to remove terms from H to construct K
- Eliminating terms at a given order in ϵ, y introduces new terms of higher order
- A term can be removed if it lies in the range of the adjoint operator of $\mathrm{ad}_{H_0} = \{\cdot, H_0\}$.
- Invoke Fredholm alternative. Resonant terms in adjoint null space. Project Hamiltonian onto this subspace.
- For example in our problem

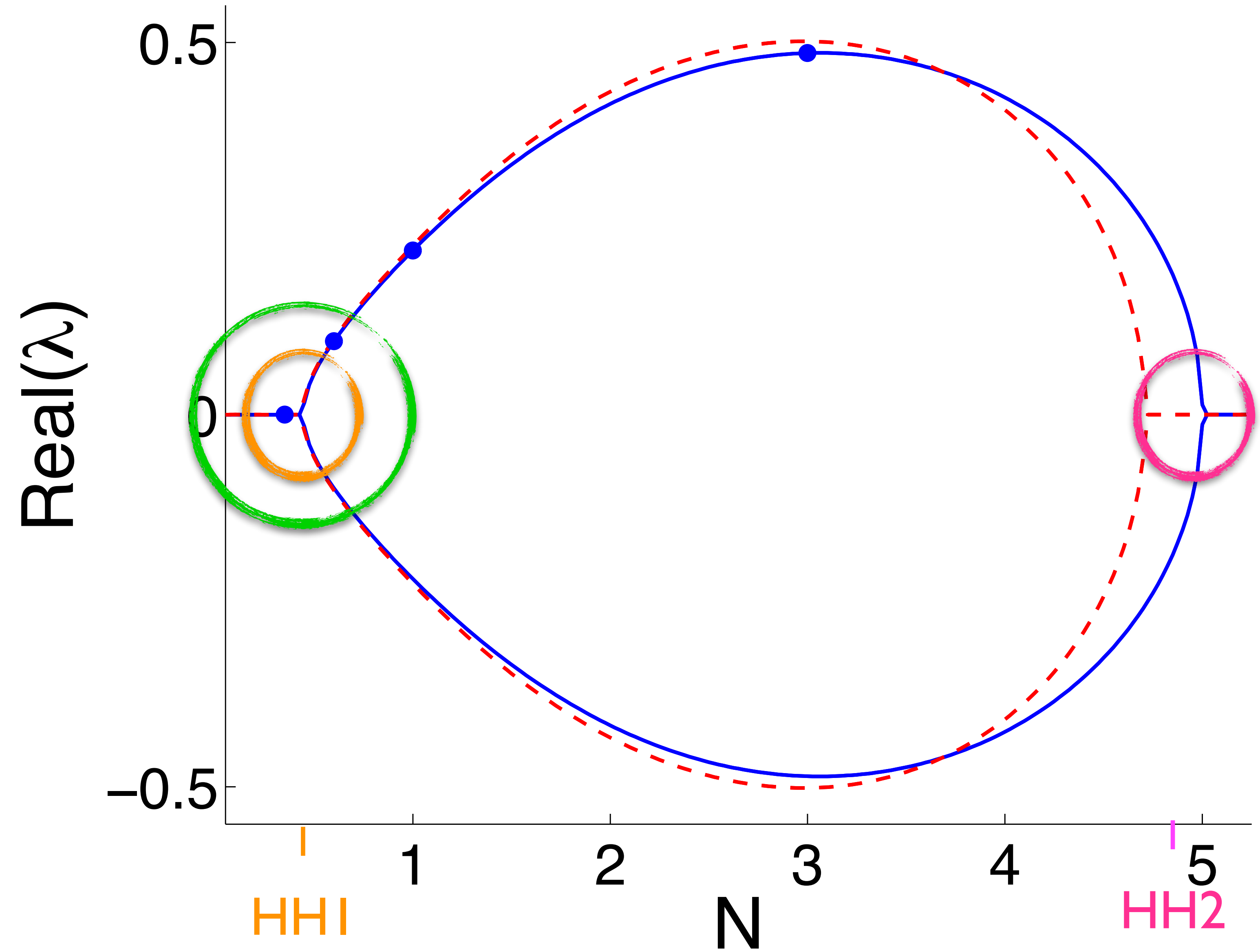
$\alpha_1 \backslash \alpha_3$	0	1
0	$\bar{z}_1 \bar{z}_3$	$\lvert z_3 \rvert^2$
1	$\lvert z_1 \rvert^2$	$z_1 z_3$

(a) Degree Two

$\alpha_1 \backslash \alpha_3$	0	1	2
0	$\bar{z}_1^2 \bar{z}_3^2$	$\lvert z_3 \rvert^2\ \bar{z}_1 \bar{z}_3$	$\lvert z_3 \rvert^4$
1	$\lvert z_1 \rvert^2\ \bar{z}_1 \bar{z}_3$	$\lvert z_1 \rvert^2\ \lvert z_3 \rvert^2$	$\lvert z_3 \rvert^2\ z_1 z_3$
2	$\lvert z_1 \rvert^4$	$\lvert z_1 \rvert^2\ z_1 z_3$	$z_1^2 z_3^2$

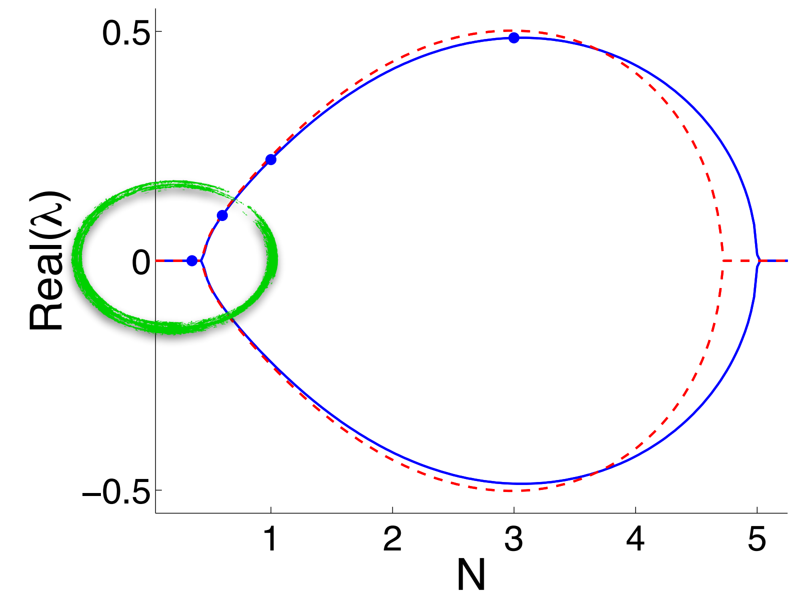
(b) Degree Four

Three normal form calculations



- Semisimple -1:1 resonance for $\epsilon \ll 1,\ N = O(\epsilon)$
  Gives HH1 at $N_{\text{crit}} = \dfrac{\epsilon}{2A} + O(\epsilon^2)$
- Nonsemisimple -1:1 resonance at N_{crit} using a further simplification of above normal form
- Nonsemisimple -1:1 resonance computed numerically at numerical location of HH2

Normal form near semisimple double eigenvalue (Chow/Kim 1988)



$$H = -\Delta|z_1|^2 + \Delta|z_3|^2$$

Normal Form

$$\begin{aligned}H_{\text{norm}} =& -\Delta|z_1|^2 + \Delta|z_3|^2 + \epsilon\left(|z_1|^2 + |z_3|^2\right) + 2AN(z_1z_3 + \bar{z}_1\bar{z}_3)\\ &+ A\left[\frac{1}{2}|z_1|^4 - 2|z_1|^2|z_3|^2 + \frac{1}{2}|z_3|^4 - 2\left(|z_1|^2 + |z_3|^2\right)(z_1z_3 + \bar{z}_1\bar{z}_3)\right]\end{aligned}$$

In Canonical Polar Coordinates

$$\begin{aligned}H =& \Delta(-J_1 + J_3) + \epsilon(J_1 + J_3) + 4AN\sqrt{J_1J_3}\cos(\theta_1 + \theta_3)\\ &+ A\left(\tfrac{1}{2}J_1^2 - 2J_1J_3 + \tfrac{1}{2}J_3^2 - 4\sqrt{J_1J_3}(J_1 + J_3)\cos(\theta_1 + \theta_3)\right)\end{aligned}$$

Independent of $(\theta_1 - \theta_3)$ implying the existence of a conserved quantity and the integrability of the Normal Form.

Advantage: Easier to find solution structure in Normal Form.

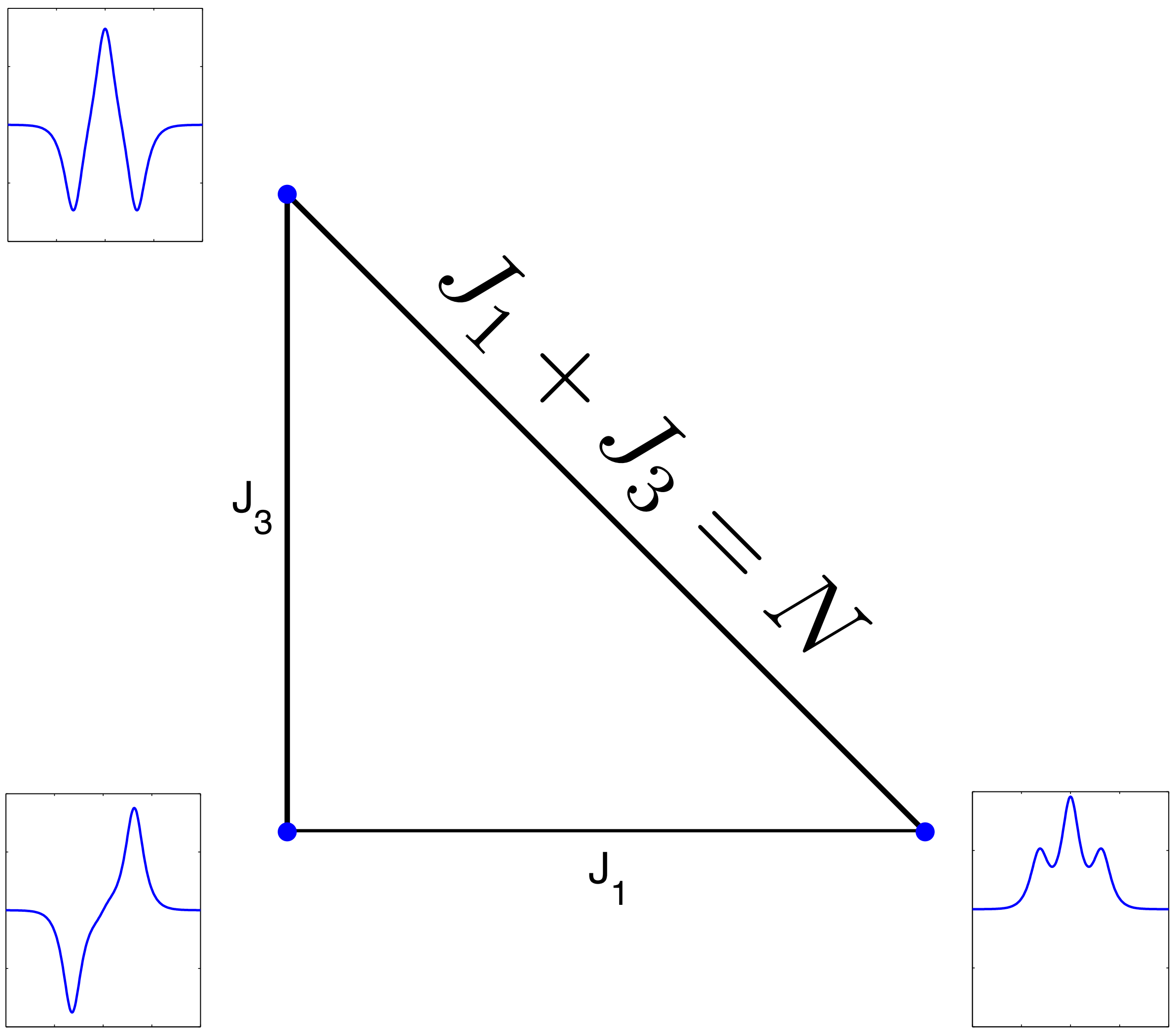
The system can be further reduced. Periodic orbits $\begin{pmatrix} J_1 \\ J_3 \end{pmatrix} e^{i\Omega t}$ solve:

$$\sqrt{J_1 J_3}\,(2\epsilon - A\,(J_1 + J_3)) + 2A\left(N\,(J_1 + J_3) - J_1^2 - 6J_1J_3 - J_3^2\right)\cos\Theta = 0$$

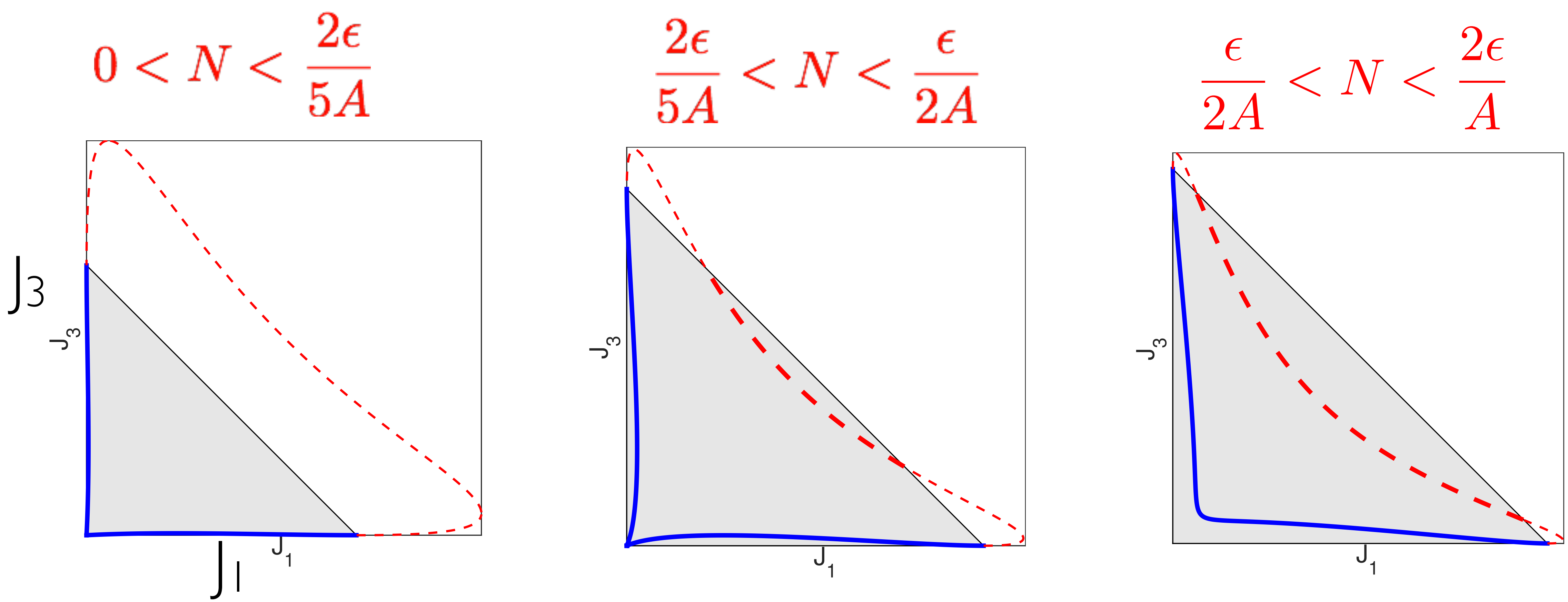
$$\sqrt{J_1 J_3}\,(N - J_1 - J_3)\sin\Theta = 0$$

With $\Theta = (\theta_1 + \theta_3)$

J_1 and J_3 act as barycentric coordinates on the triangle of admissible solutions showing relative strength of the three modes.



Sequence of bifurcations in Normal Form



2 Lyapunov families of fixed points + unphysical branch

Unphysical branches cross into physical region

Lyapunov branches "pinch off"

Question: At second bifurcation point HH2, must have Lyapunov families of fixed point. Where do they come from?

Normal form for *non*-semisimple -1:1 resonances at HH1 and HH2 (Meyer-Schmidt 1974)

In symplectic polar coordinates $(r, \theta, p_r, p_\theta)$, this is:

$$H = H_0(r, p_r, p_\theta) + \mu^2 \delta H_2(r, p_\theta) + H_4(r, p_\theta)$$

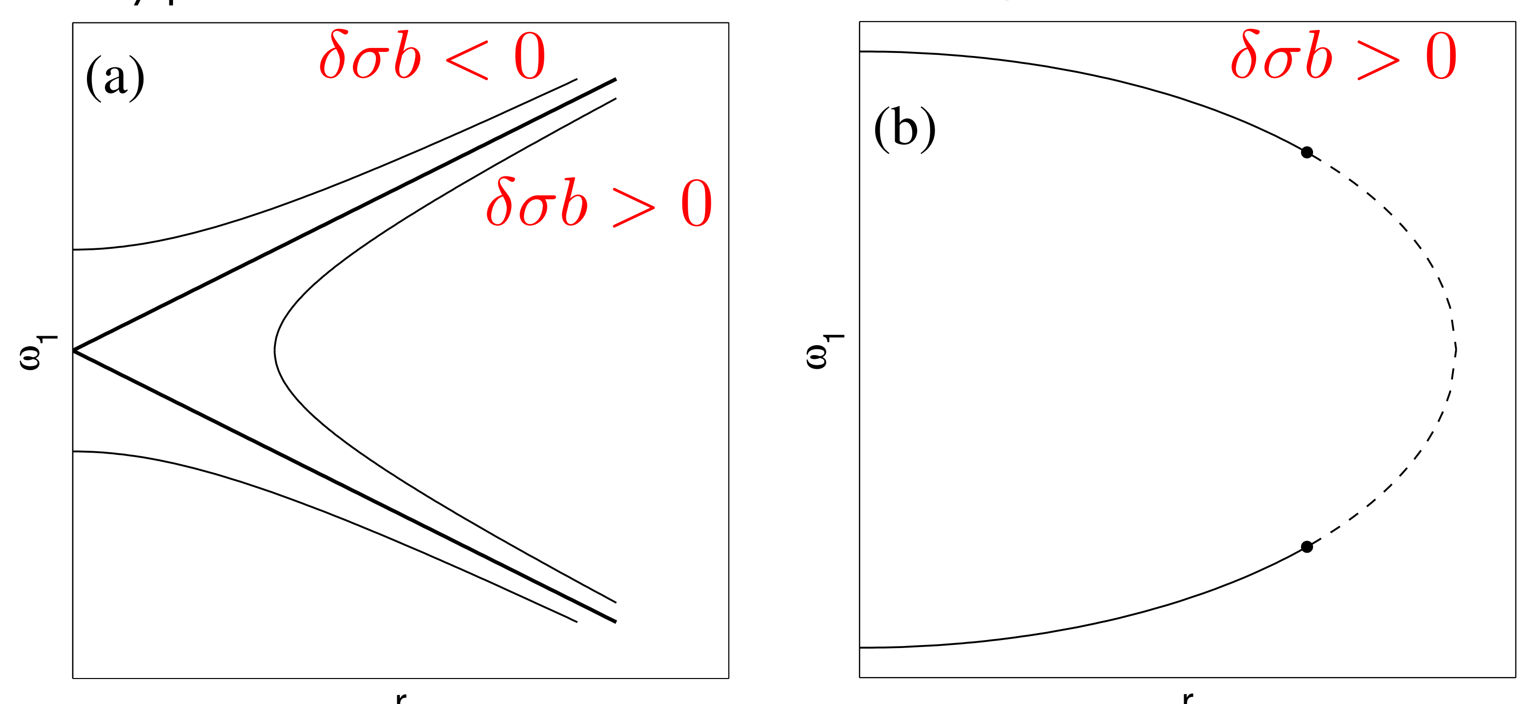
$$= \Omega p_\theta + \frac{\sigma}{2}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) + \mu^2 \delta \left(a p_\theta + \frac{b}{2} r^2\right) + \frac{c}{2} p_\theta^2 + \frac{d}{2} p_\theta r^2 + \frac{e}{8} r^4$$

$\delta = \pm 1,\ \mu \ll 1$

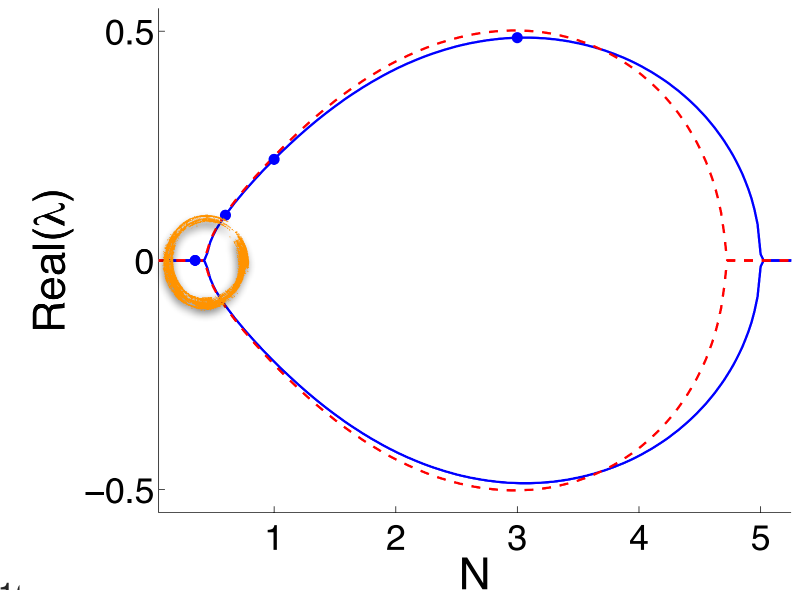
Poincaré-Lindstedt argument: periodic orbits with "amplitude" μr and frequency $\Omega + \mu\omega_1$ when there is a solution to $2\omega_1^2 - \sigma e r^2 = 2\delta\sigma\beta$

Two cases:

Hyperbolic $\sigma e > 0$ | Elliptic $\sigma e < 0$

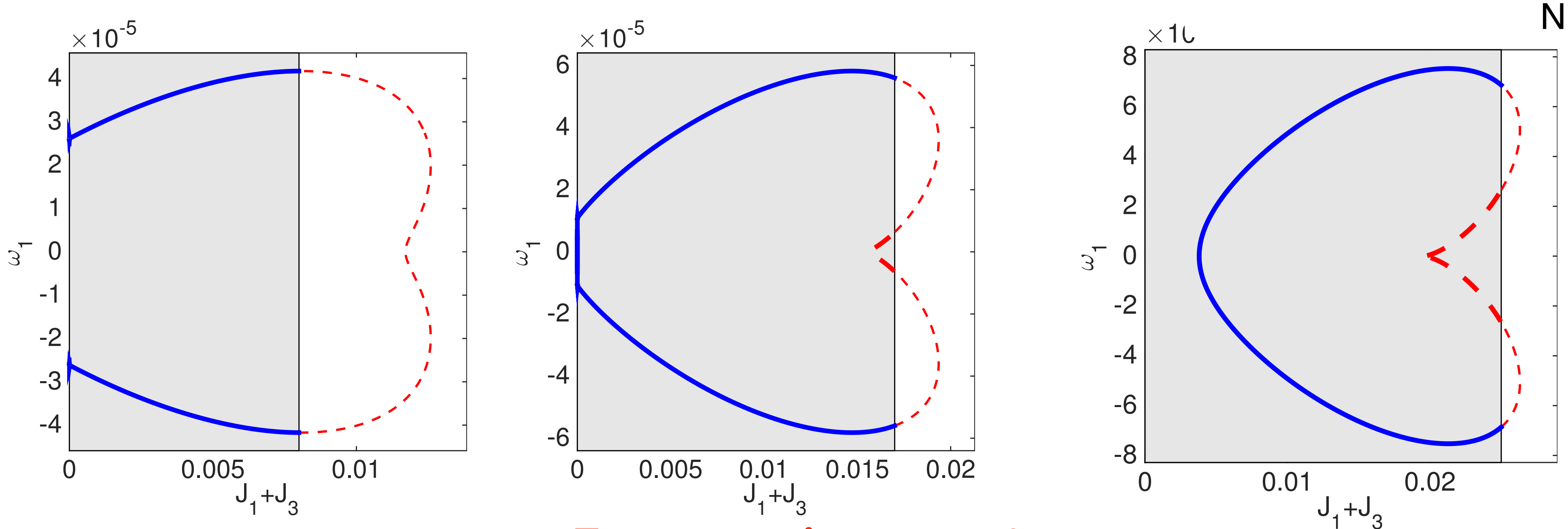


The bifurcation at HH1



Computations using previous normal form

Frequency

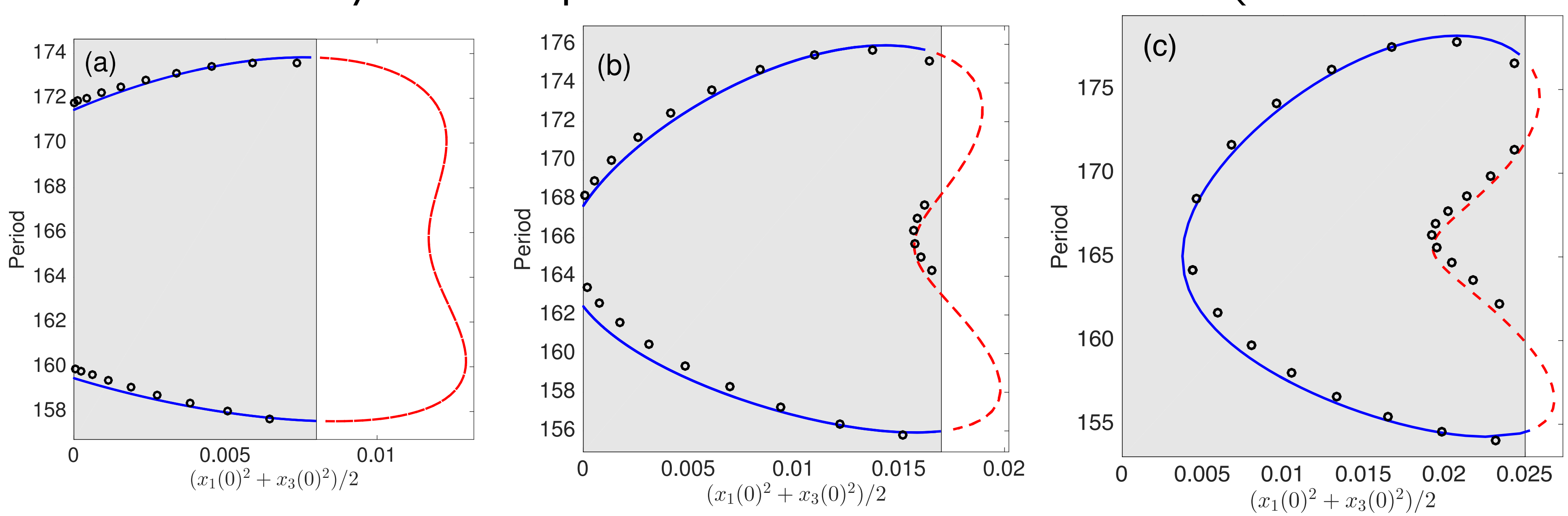


"Amplitude"

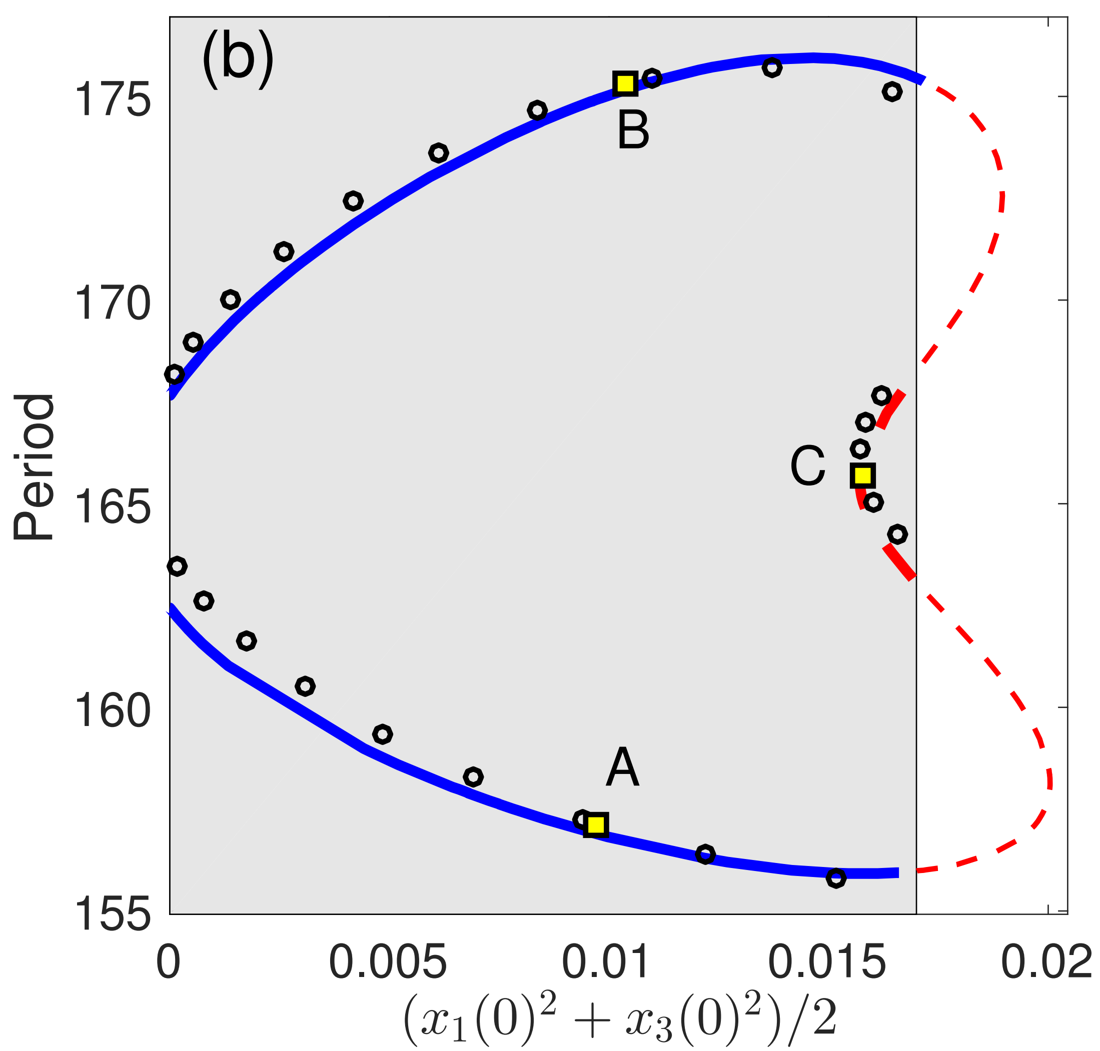
Increasing *N*➔

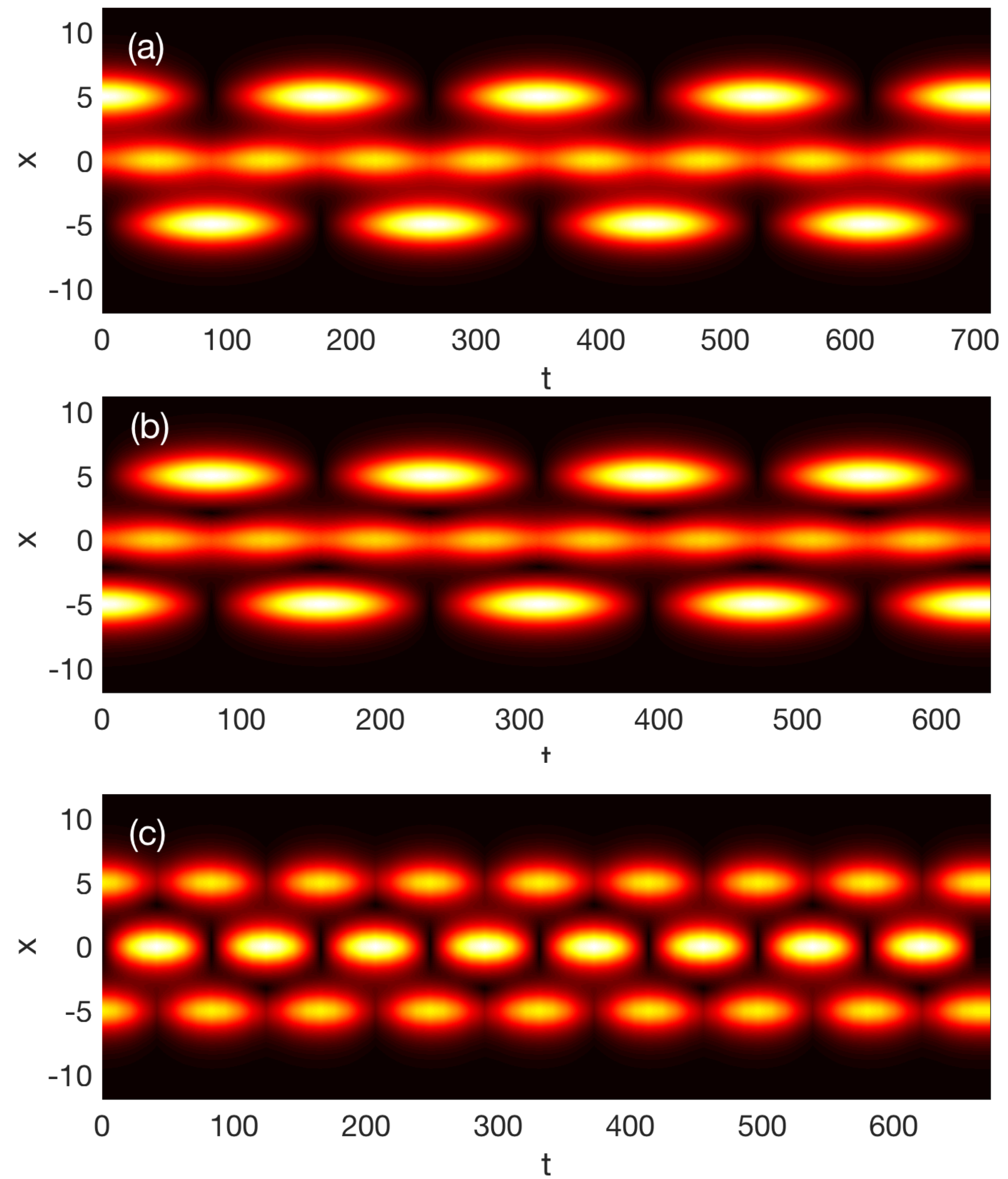
Numerically Computed Periodic orbits (not normal form)

Period

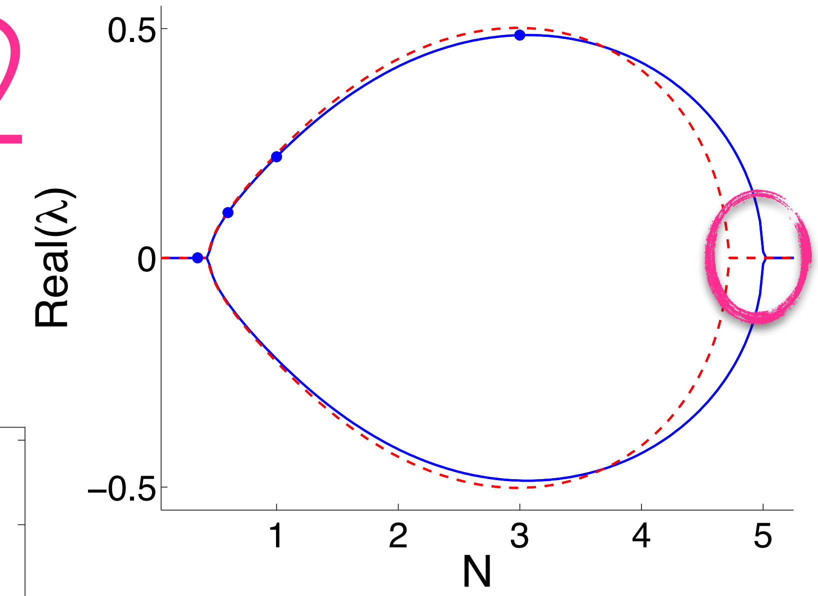


Some computed PDE solutions on this branch



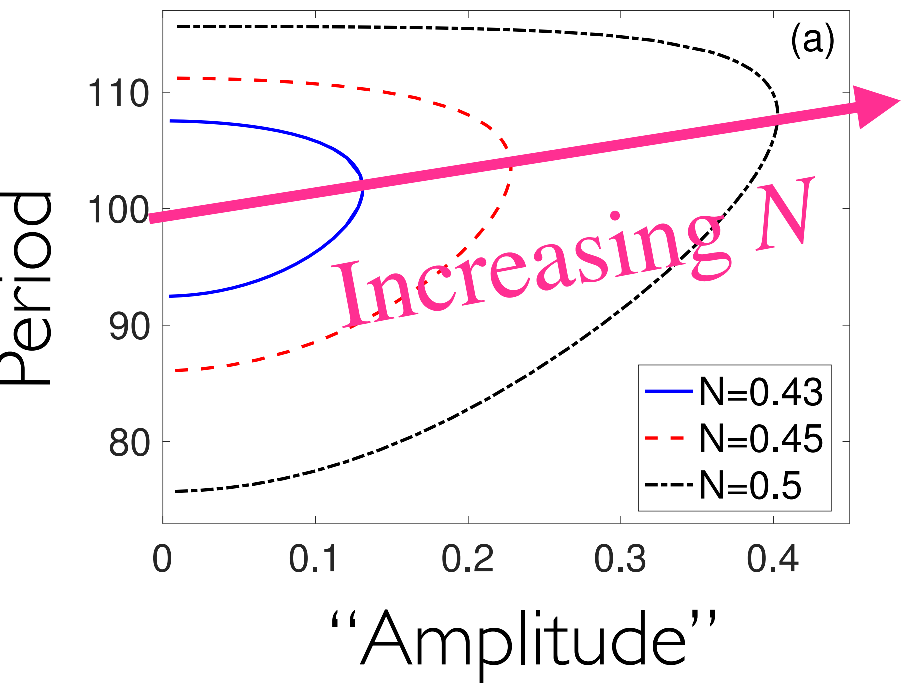


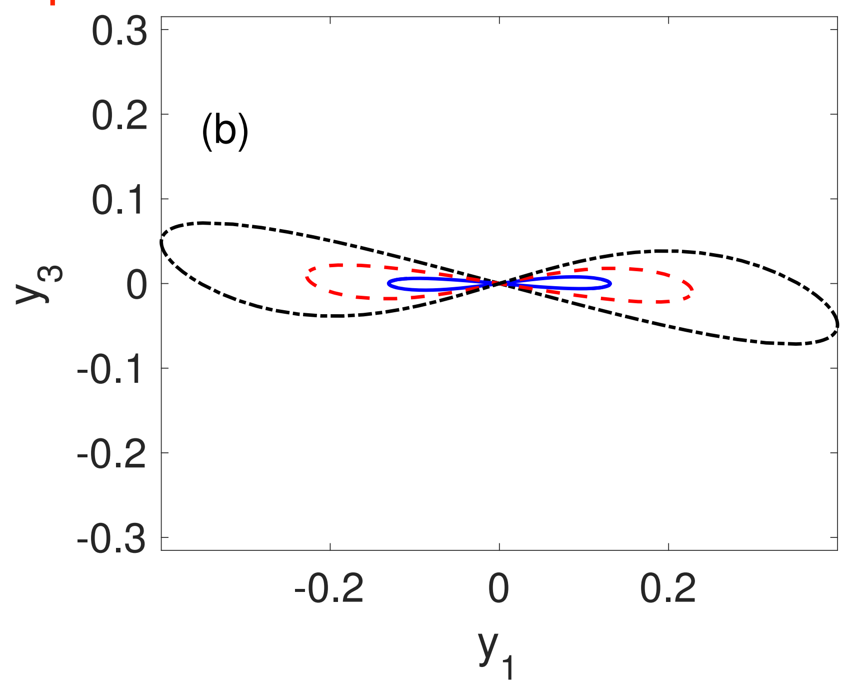
The bifurcation at HH2



Numerically Computed Periodic orbits

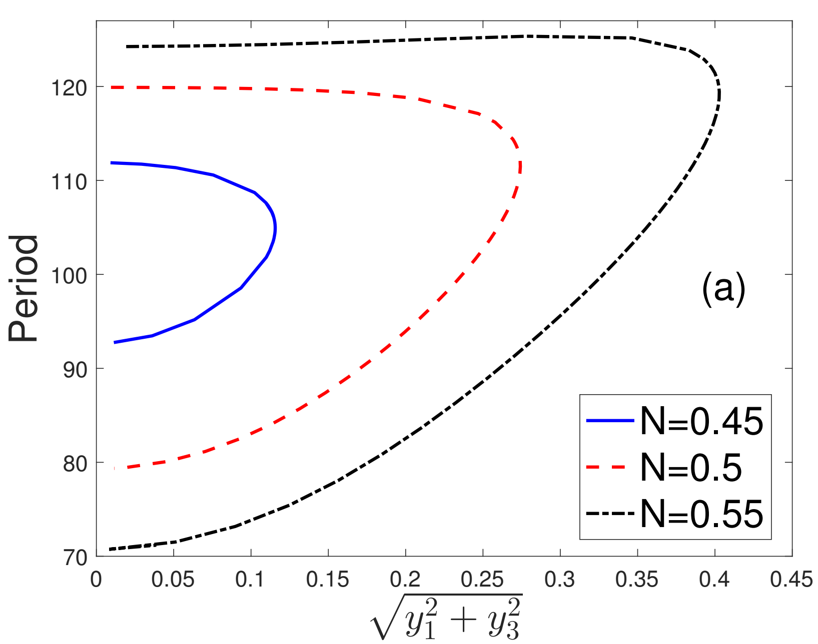
ODE Computation

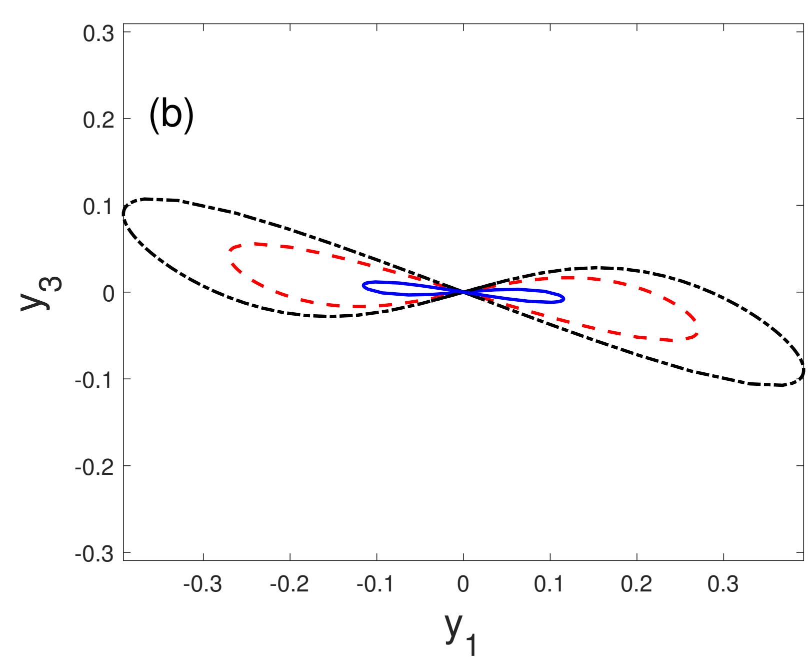


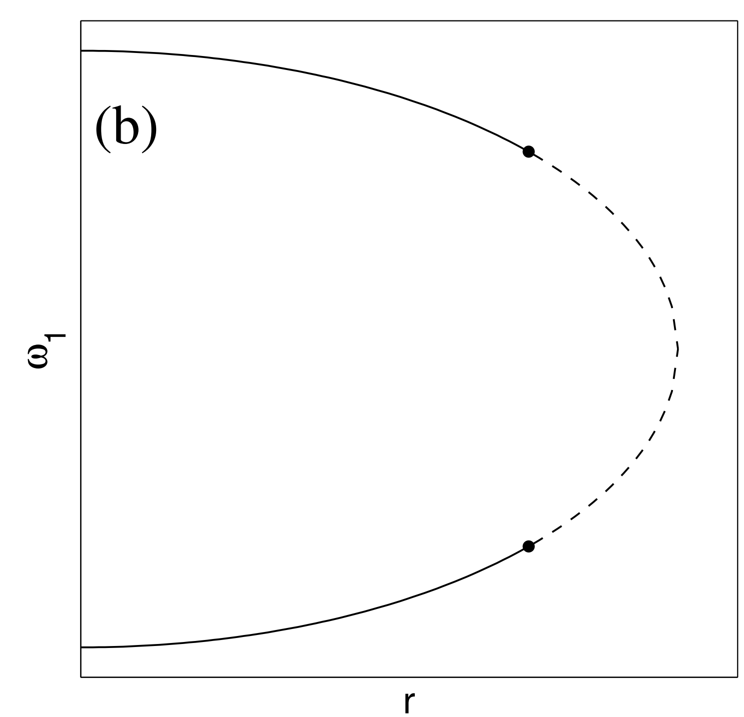


New family of periodic orbits arises in "elliptic" HH bifurcation

PDE Computation







Increasing N

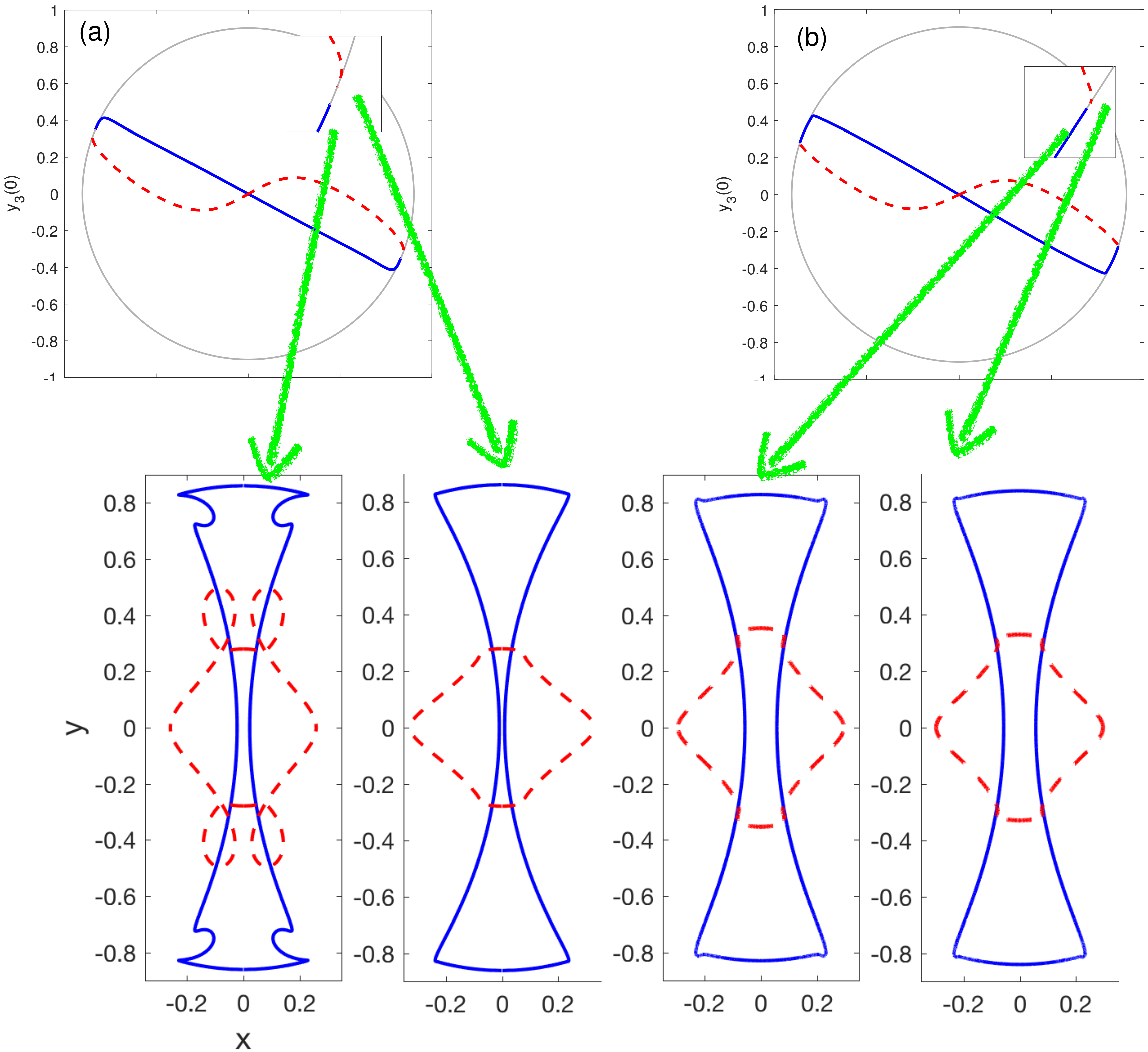
Solutions must satisfy $|z_1|^2 + |z_3|^2 < N$.

PDE

$N = 0.82$

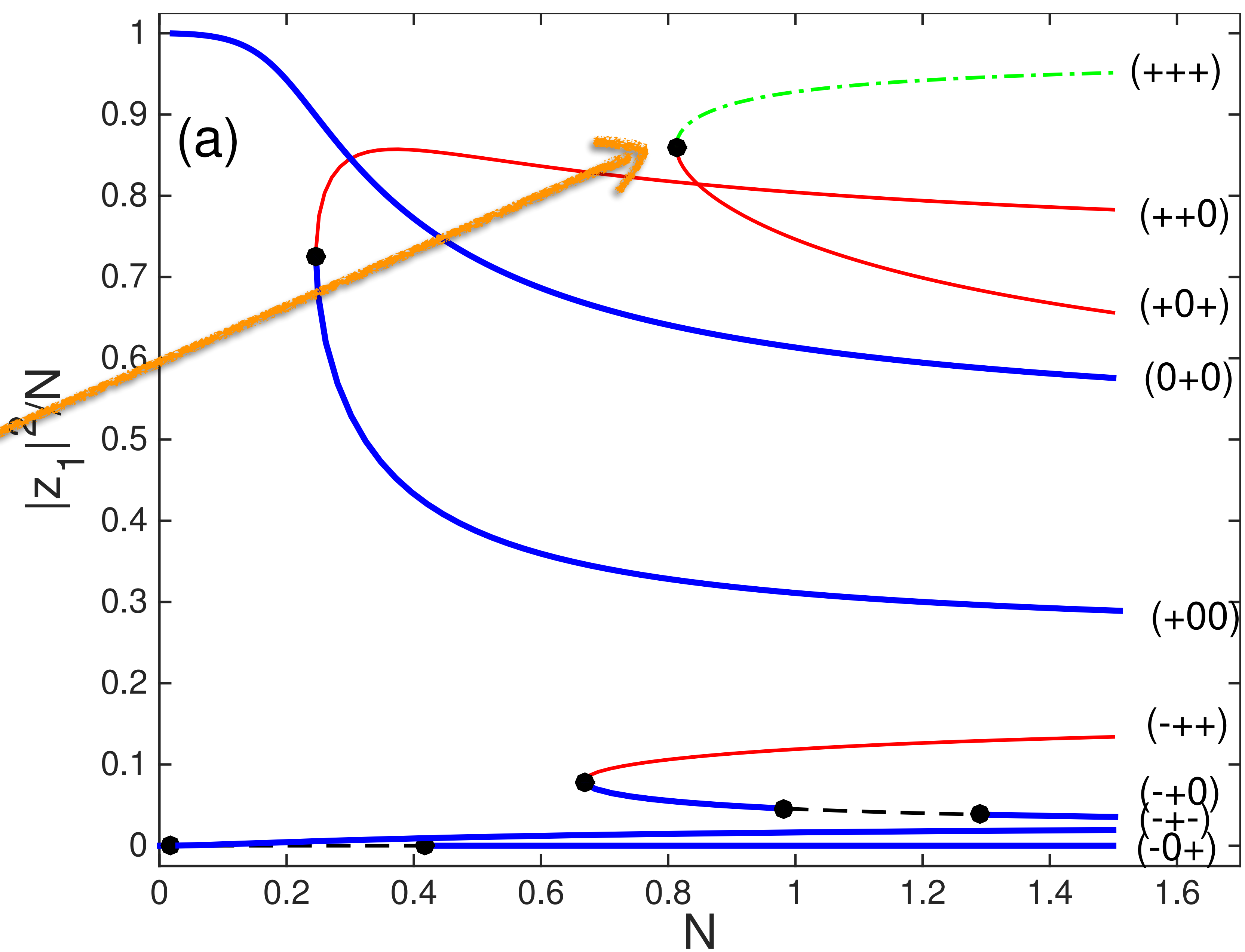
ODE

$N = 0.8135$



What's going on?

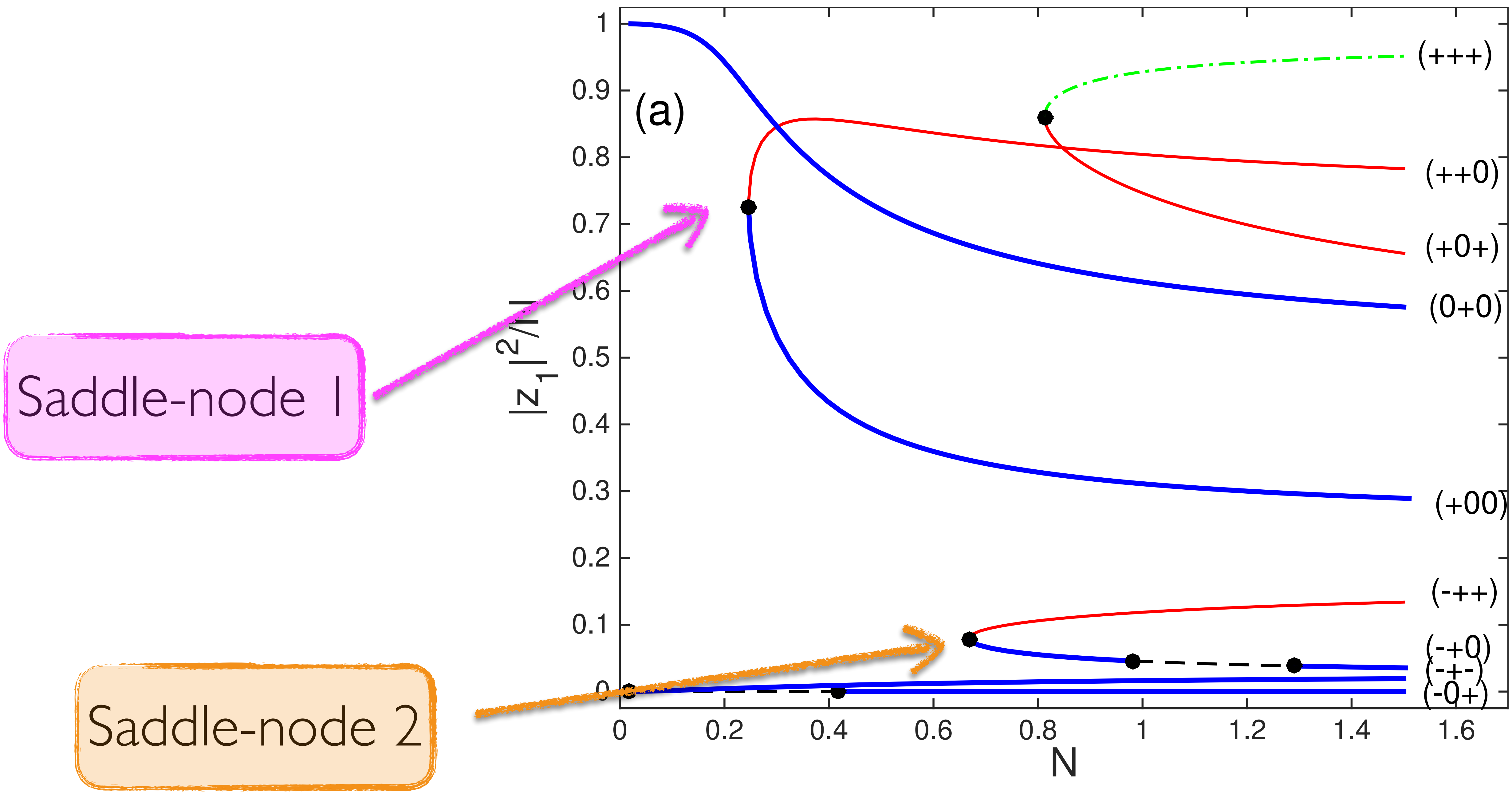
Getting close to other fixed points



$N = 0.8125$

What about the other Lyapunov branches of periodic orbits?

I thought saddle-node bifurcations were boring



Normal form for $0^2 i\omega$ bifurcation

Small beyond all orders remainder

$$H = \left(\frac{q_1^2}{2} + \frac{p_1^2}{2}\right) + \alpha\left(\frac{p_2^2}{2} + \delta q_2 - \frac{q_2^3}{3}\right) + \beta I q_2 + H_{\text{higher}}(q_2, I) + R_\infty(q_1, q_2, p_1, p_2)$$

Fast & Oscillatory

Saddle node bifurcation at $\delta = 0$

Coupling

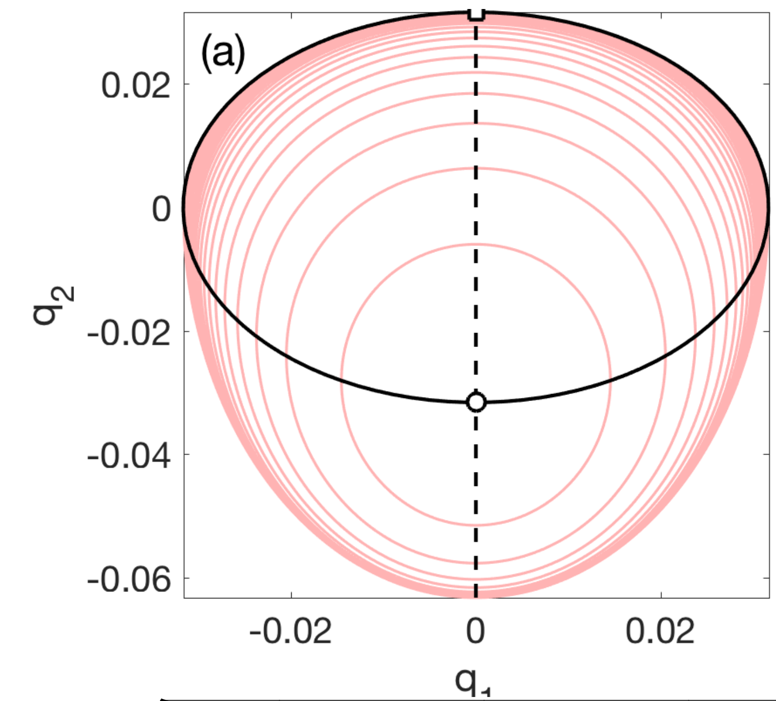
where $I = \left(\frac{q_1^2}{2} + \frac{p_1^2}{2}\right)$

Three families of periodic orbits:

- Fast
- Slow
- Mixed

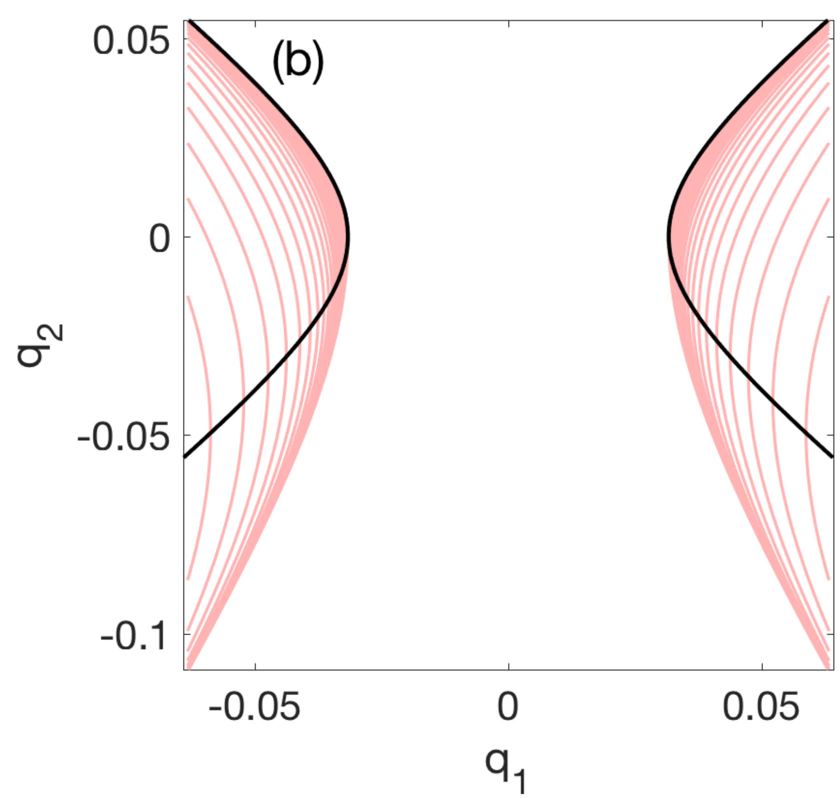
Perturbation expansion shows two regimes

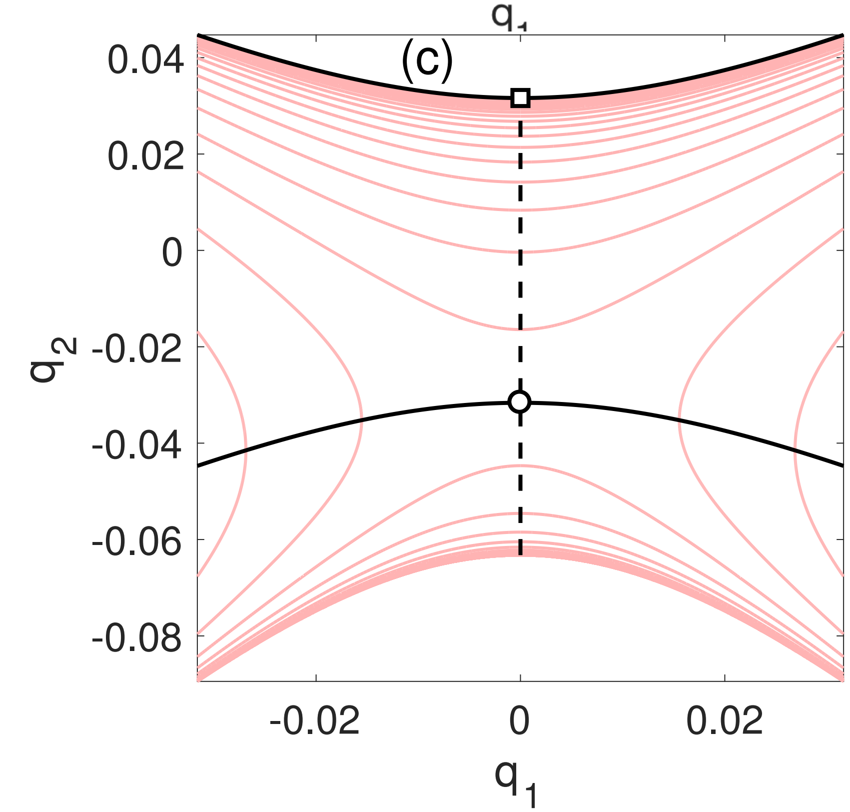
$N < N_{\mathrm{crit}}$ $N > N_{\mathrm{crit}}$



Saddle-node 1

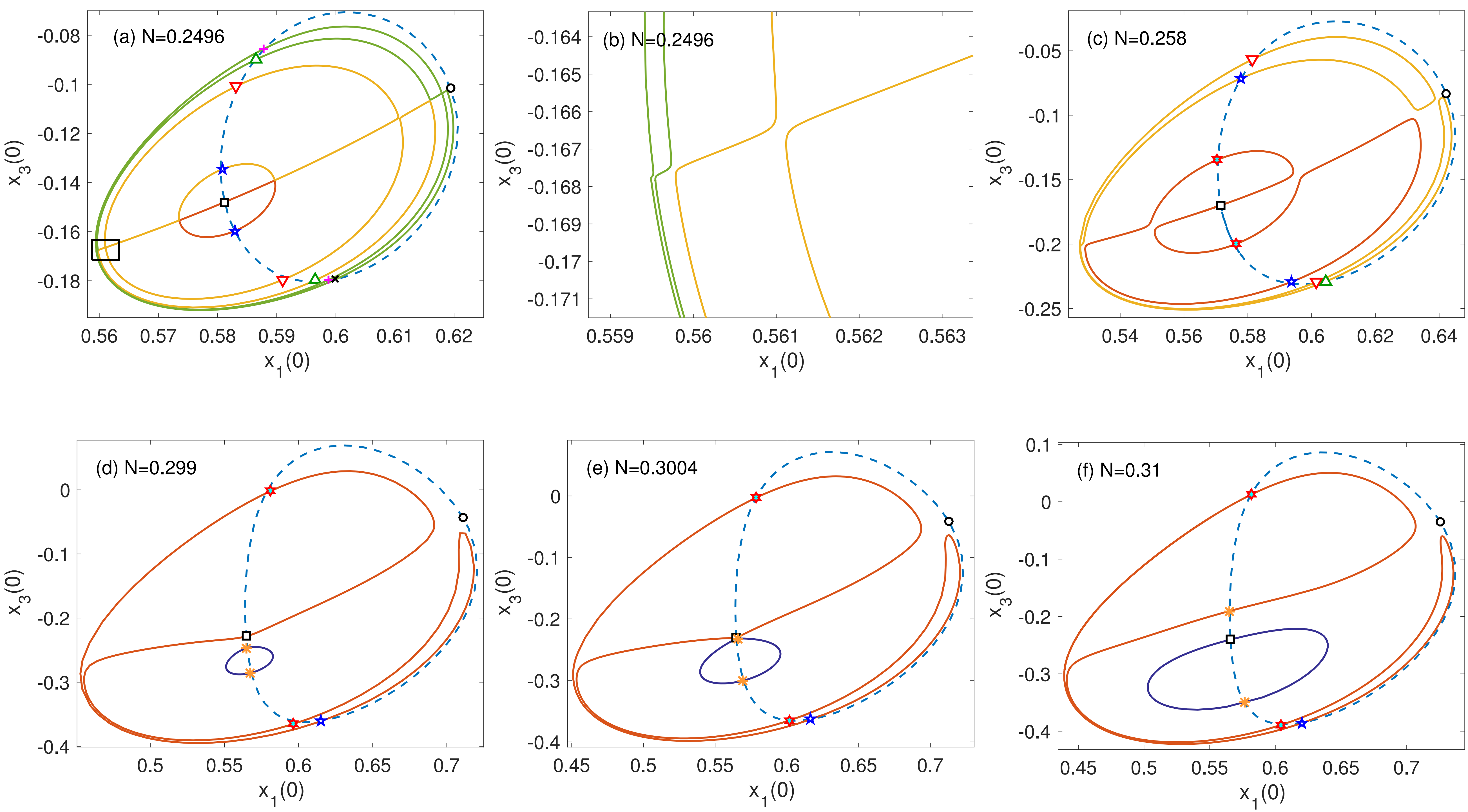
Saddle-node 2





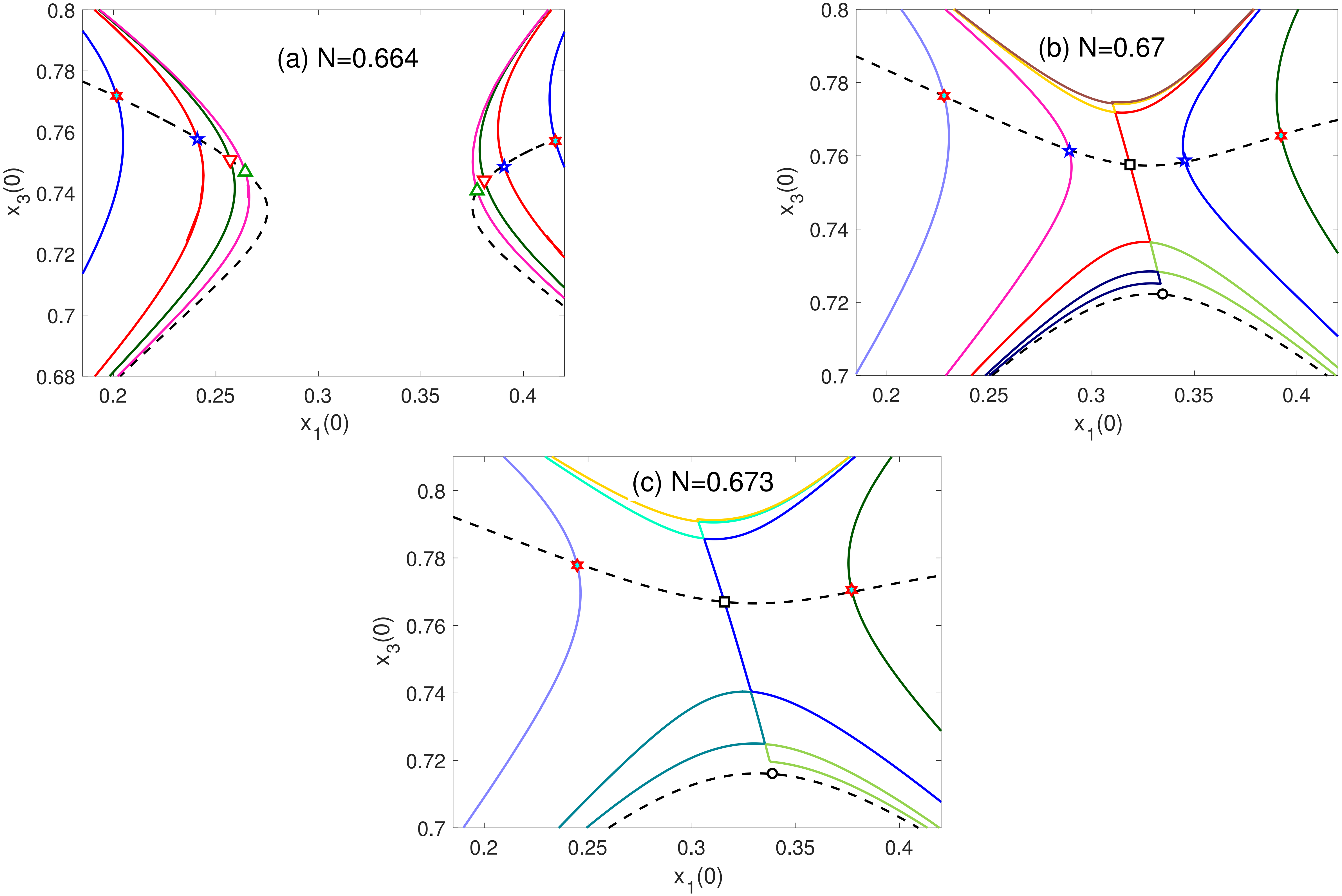
Mixed periodic orbits bifurcate when $\delta = \dfrac{1}{\alpha^4 n^4}$

Saddle-node I



Gelfreich-Lerman 2003

Saddle-node 2



Parting Words

- This problem has an ODE part and a PDE part
- Increasing from two wells to three makes the ODE part of the problem hard
- In addition to standing waves, there is a whole lot of additional structure in solutions that oscillate among the three waveguides
- Normal forms give us a partial picture of the reduced dynamics
- Even saddle-node bifurcations are interesting.
- Big question: What can be proven about shadowing these orbits in NLS/GP?

For re/preprints http://web.njit.edu/~goodman

Thanks!