Leapfrogging Vortex Pairs

Linear Stability, Nonlinear Dynamics, & Escape

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CMS Winter Meeting, Toronto

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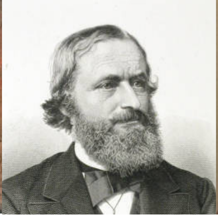
Dramatis Personae

Act I: 19th Century

Helmholtz



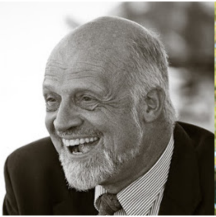
Kirchhoff

Gröbli

A. E. H. Love

Act II: Late 20th–Early 21st Century

Acheson



Aref

Kevrekidis

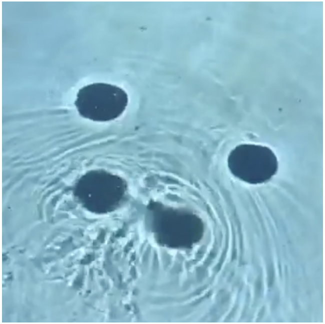
Behring

Leapfrogging Vortex Rings

Helmholtz (1858):

> *The foremost widens and travels more slowly, the pursuer shrinks and travels faster, till finally, if their velocities are not too different, it overtakes the first and penetrates it. Then the same game goes on in the opposite order, so that the rings pass through each other alternately.*

Credit: Irvine Lab, University of Chicago



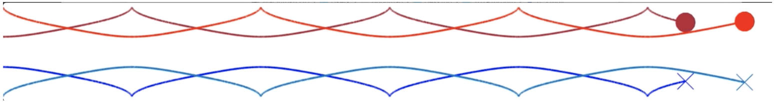
Credit: thephysicsgirl on Instagram

Leapfrogging quartets of point vortices

- Cutting a concentric pair of vortex rings along a diameter gives a quartet of vortices: a simplified model of vortex rings.

Leapfrogging quartets of point vortices

- Cutting a concentric pair of vortex rings along a diameter gives a quartet of vortices: a simplified model of vortex rings.
- Gröbli (1877) and Love (1883) independently discovered and analyzed a one-parameter family of four point-vortex orbits:



Helmholtz derivation of the vortex induction equations

- Let $\mathbf{u}(\mathbf{x}, t)$ solve the 2D Euler's equation.
- Particles advected according to $\dot{\mathbf{x}} = \mathbf{u}(\mathbf{x}, t)$.
- By Helmholtz decomposition

$$\mathbf{u} = \nabla\phi + \nabla \times \psi$$

  where $\triangle\psi = -\omega$ and $\omega = \nabla \times \mathbf{u}$.
- Let vorticity be concentrated at N points $\mathbf{x}_i$ of circulation Γ_i:

$$\omega(\mathbf{x}) = \sum_{i=1}^{N} \Gamma_i \delta(\mathbf{x} - \mathbf{x_i}).$$

- Velocity due to each vortex given by the Green's function for 2D Poisson equation, yielding evolution equations:

$$\dot{x}_i = -\frac{1}{2\pi} \sum_{j \neq i}^{N} \Gamma_j \frac{(y_i - y_j)}{||\mathbf{x}_j - \mathbf{x}_i||^2} \quad \text{and} \quad \dot{y}_i = +\frac{1}{2\pi} \sum_{j \neq i}^{N} \Gamma_j \frac{(x_i - x_j)}{||\mathbf{x}_j - \mathbf{x}_i||^2}.$$

Kirchhoff's Hamiltonian Formulation

Define complex position coordinates

$$z_j(t) = x_j + iy_j$$

and Hamiltonian

$$H = -\frac{1}{2\pi} \sum_{1 \le i < j \le N} \Gamma_i \Gamma_j \log \left| z_i - z_j \right|.$$

This gives rise to a system of 2N *first order* equations-of-motion

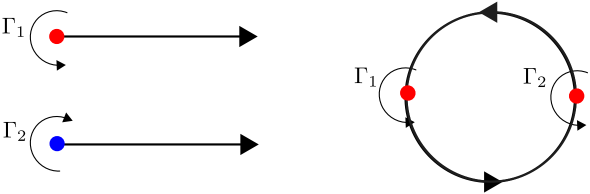
$$\Gamma_j \dot{z}_j = -2i \frac{\partial H}{\partial \bar{z}_j}.$$

The components x_i and y_i are conjugates: hase space coincides with configuration space.

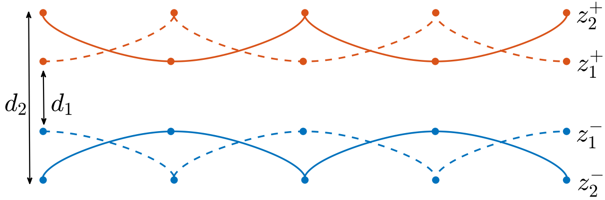
Building up to it: two vortices

- Opposite-signed vortices move in parallel along straight lines.
- Like-signed vortices move in a circular path with a constant rotation rate.

(a) $\Gamma_1 = -\Gamma_2 > 0$ (b) $\Gamma_1 = \Gamma_2 > 0$



Schematic of the leapfrogging solution



Solutions form a one-parameter family of relative periodic orbits for parameter values $\alpha = \frac{d_1}{d_2}$ for $3 - 2\sqrt{2} \approx 0.171 < \alpha < 1$.

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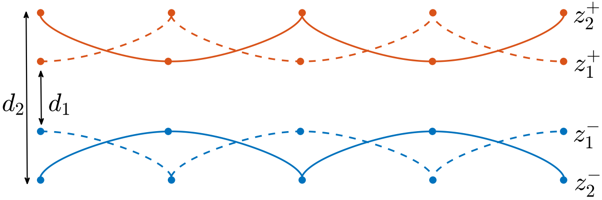

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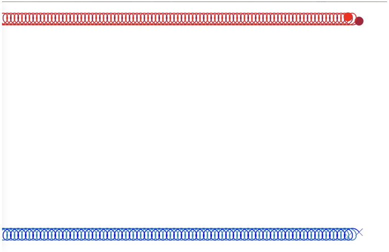

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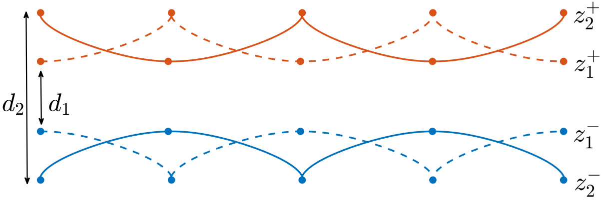

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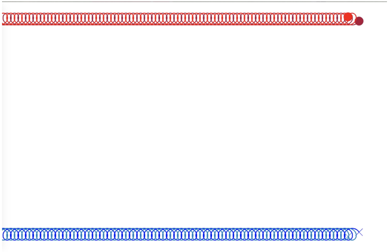

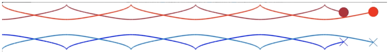

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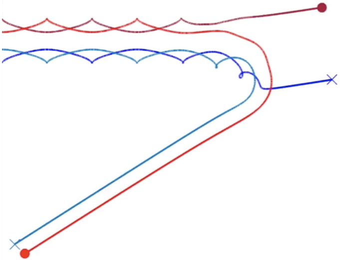

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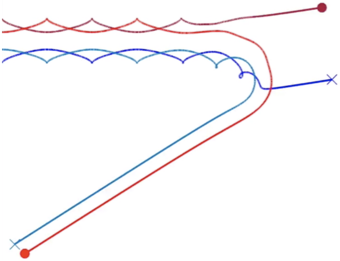
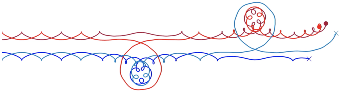
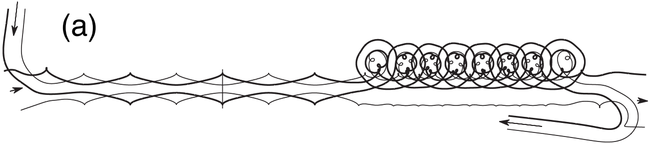


For $0.29 < \alpha < 0.382$, motion goes into *walkabout* orbit:



Prior Results: Tophøj & Aref (2013) *Phys. Fluids*

- $\alpha_c = 0.382 = \frac{1}{\phi^2}$ to many digits, where ϕ is the golden ratio. Shown by numerical solution of linearized problem.
- No clean distinction between domains of disintegration and walkabout solutions



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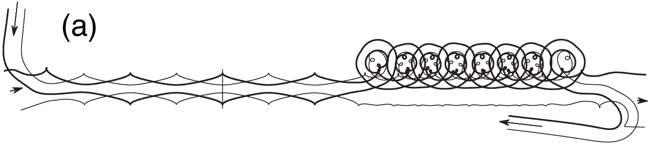

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- Derive the linear stability threshold
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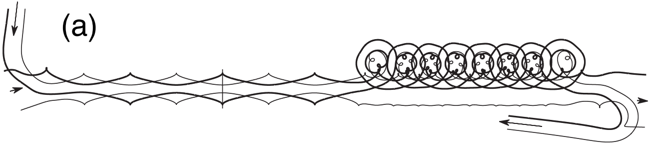

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These require two different coordinate systems

The fundamental question

Why does the bifurcation take place at specific value

$$\alpha_c = \frac{1}{\phi^2}?$$

Reference:

B. M. Behring and R. H. Goodman. Stability of leapfrogging vortex pairs: A semi-analytic approach. To appear in *Phys. Rev. Fluids* `https://arxiv.org/abs/1908.08618`, 2019.

Reducing the Hamiltonian

$$\mathcal{H} = \frac{1}{4\pi}\Big(- \log|z_2^- - z_1^-|^2 - \log|z_1^+ - z_2^+|^2 + \log|z_1^+ - z_1^-|^2 + \log|z_2^+ - z_1^-|^2 + \log|z_1^+ - z_2^-|^2 + \log|z_2^+ - z_2^-|^2\Big).$$

- Introduce mean-and-difference coordinates:

$$z_+ = \tfrac{1}{2}\left(z_1^+ + z_2^+\right), \; z_- = \tfrac{1}{2}\left(z_1^- + z_2^-\right), \; \delta_+ = z_1^+ - z_2^+, \;\; \delta_- = z_1^- - z_2^-$$

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- WLOG, take $M = 2i$. Using these two conserved quantities, reduce to 2 degrees of freedom *Dimer coordinates*
- One more change yields *Aref coordinates* which he gave as a *complex Hamiltonian:*

$$\tilde{\mathcal{H}}(Z, W) = -\frac{1}{2}\log\left(\frac{1}{1+Z^2} - \frac{1}{1+W^2}\right),$$

$Z = X + iP$ and $W = Q + iY$, with conjugate pairs (X, Y) and (Q, P).

Regularizing the leapfrogging orbits

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- Nonlinearly rescale the Hamiltonian

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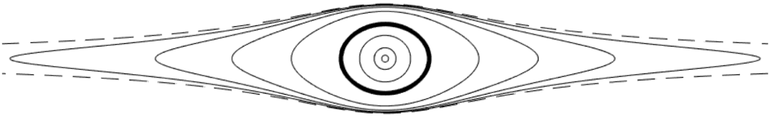
- Introduce energy level $h = \frac{(1-\alpha)^2}{8\alpha}$ as the new parameter.
- Periodic orbits exist for $0 \le h < \frac{1}{2}$ and are stable for $0 \le h \le h_c = \frac{1}{8}$.

The leapfrogging solutions in Aref Coordinates

- The (X, Y) phase plane:



- Gröbli found an exact *implicit* solution

$$t(X) = \frac{1}{2h^2\sqrt{1-4h^2}} F\left(\sin^{-1}\theta \middle| k\right) - E\left(\sin^{-1}\theta \middle| k\right) - \frac{1+2h}{2h\sqrt{(1-2h)\left(2h\left(X^2+1\right)+1\right)}}.$$

where $\theta = X\sqrt{\frac{2h-1}{2h}}$, $k^2 = \frac{4h^2}{4h^2-1}$.

Linearized equations

We introduce perturbation coordinates

$$Z(t) = X(t) + [\xi_+(t) + i\eta_+(t)],$$
$$W(t) = iY(t) + [\xi_-(t) + i\eta_-(t)],$$

yielding linearized equations that decouple into:

$$\frac{d}{dt}\left[\xi_+, \eta_-\right]^{\mathsf{T}} = A^{\mathsf{T}}(X,Y)\left[\xi_+, \eta_-\right]^{\mathsf{T}},$$
$$\frac{d}{dt}\left[\xi_-, \eta_+\right]^{\mathsf{T}} = A(X,Y)\left[\xi_-, \eta_+\right]^{\mathsf{T}},$$

where

$$A(X(t),Y(t);h) = \begin{pmatrix} \frac{XY}{(X^2+Y^2)(1+X^2)(1-Y^2)} & -\frac{3Y^4+X^2Y^2+X^2-Y^2}{2(X^2+Y^2)(1-Y^2)^3} \\ -\frac{3X^4+X^2Y^2-Y^2+X^2}{2(X^2+Y^2)(1+X^2)^3} & -\frac{XY}{(X^2+Y^2)(1+X^2)(1-Y^2)} \end{pmatrix}.$$

The 1st ODE governs perturbation in the invariant plane and is stable. The 2nd governs stability.

Obtaining explicit linearized equations

Problem: In the expression $\frac{d}{dt}Z = A(X(t), Y; h)Z$, $(X(t), Y(t))$ are only known implicitly. To resolve:

- Rewrite $A(X, Y; h)$ in terms of canonical polar variables

$$X = \sqrt{2J}\cos\theta, Y = \sqrt{2J}\sin\theta.$$

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$$\frac{d}{dt} = \frac{d\theta}{dt}\frac{d}{d\theta} = \frac{dH}{dJ}\frac{d}{d\theta}.$$
- Then
$$\frac{d}{d\theta}Z(\theta) = \tilde{A}_h(\theta)Z(\theta) \text{ where } \tilde{A}_h(\theta) = \left(\left.\frac{dH}{dJ}\right|_{H=h}\right)^{-1} A(\theta, h).$$

The linearized problem at $h = \frac{1}{8}$

- When $h = 1/8$, the relevant linearized equation is

$$\tilde{A}_{\frac{1}{8}}(\theta) = \frac{1}{4\sqrt{17+8\cos 2\theta}} \times \begin{pmatrix} -\sin 2\theta & \frac{7+12\cos 2\theta - 4\cos 4\theta - 3\sqrt{17+8\cos 2\theta}}{2-2\cos 2\theta} \\ \frac{3-4\cos 2\theta - 4\cos 4\theta - \sqrt{17+8\cos 2\theta}}{2+2\cos 2\theta} & \sin 2\theta \end{pmatrix}.$$

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- *Floquet theory in 3 lines:*
  - For $h < h_c$ Floquet multipliers on unit circle: linearized orbits quasiperiodic.
  - For $h > h_c$ real, reciprocal Floquet multipliers: linearized orbits grow or decay.
  - For $h = h_c$, double unit Floquet multiplier: the linearized system has an orbit of period 2π.

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  - For $h = h_c$, double unit Floquet multiplier: the linearized system has an orbit of period 2π.
- *Numerical evidence:* Simulation with 30^{th}-order ODE solver & high-precision arithmetic shows that $\vec{Z}(2\pi)$ is within 10^{-120} of $Z(0)$ when $h = \frac{1}{8}$.

The method of harmonic balance (Hill, Poincaré)

- Consider an ODE

$$\frac{d}{dt}\vec{x} = \tilde{A}_h(\theta)\vec{x} = \sum_{n=0}^{\infty} h^n A_n(\theta)\vec{x}$$

  where $h \ll 1$ and $A_n(\theta + 2\pi) = A_n(\theta)$.

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where $h \ll 1$ and $A_n(\theta + 2\pi) = A_n(\theta)$.
- Looking for periodic solutions of the form
$$\vec{x} = \left(\sum_{n=0}^{\infty} \alpha_n \cos n\theta, \sum_{n=1}^{\infty} \beta_n \sin n\theta\right)^T,$$
derive a countable system of algebraic equations for $(\vec{\alpha}, \vec{\beta})$.

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$$\frac{d}{dt}\vec{x} = \tilde{A}_h(\theta)\vec{x} = \sum_{n=0}^{\infty} h^n A_n(\theta)\vec{x}$$

where $h \ll 1$ and $A_n(\theta + 2\pi) = A_n(\theta)$.

- Looking for periodic solutions of the form

$$\vec{x} = \left(\sum_{n=0}^{\infty} \alpha_n \cos n\theta, \sum_{n=1}^{\infty} \beta_n \sin n\theta \right)^T,$$

derive a countable system of algebraic equations for $(\vec{\alpha}, \vec{\beta})$.

- Truncating in both h and in Fourier space yields a sequence of finite-dimensional linear equations

$$M^{(N)}(h) \begin{pmatrix} \vec{\alpha}_N \\ \vec{\beta}_N \end{pmatrix} = 0.$$

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$$M^{(N)}(h) \begin{pmatrix} \vec{\alpha}_N \\ \vec{\beta}_N \end{pmatrix} = 0.$$

- Solving $\det M^{(N)}(h) = 0$ yields a sequence of polynomials whose roots approximate h at which periodic orbits exist.

Harmonic balance, applied

First few terms in expansion:

$$A_0(\theta) = \begin{pmatrix} -\sin 2\theta & -\cos 2\theta \\ -\cos 2\theta & \sin 2\theta \end{pmatrix}, A_1(\theta) = \begin{pmatrix} \sin 4\theta & 3+\cos 4\theta \\ 3+\cos 4\theta & -\sin 4\theta \end{pmatrix},$$

$$A_2(\theta) = \frac{1}{2}\begin{pmatrix} \sin 2\theta - 3\sin 6\theta & -12 - 9\cos 2\theta - 3\cos 6\theta \\ 12 + 9\cos 2\theta - 3\cos 6\theta & -\sin 2\theta + 3\sin 6\theta \end{pmatrix}.$$

... after many, many implementation details ...

$$\left|M^{(1)}\right| = \begin{vmatrix} -1+h & 2+2h \\ -2h & 1-h \end{vmatrix} = -1 + 6h + 3h^2,$$

$$\left|M^{(2)}\right| = \begin{vmatrix} -1+h & 2+2h+8h^2 & -h-\frac{h^2}{2} & -2h-2h^2 \\ -2h-4h^2 & 1-h & -2h+2h^2 & -h+\frac{h^2}{2} \\ h-\frac{h^2}{2} & -2h-2h^2 & -3 & 2+8h^2 \\ -2h+2h^2 & h+\frac{h^2}{2} & -4h^2 & 3 \end{vmatrix}$$

$$= 9 - 54h - 109h^2 - 210h^3 - \frac{977h^4}{2} + \frac{1049h^5}{2} + \frac{75h^6}{2} + 1074h^7 + \frac{11233h^8}{16}.$$

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Implement to arbitrary order in Mathematica:

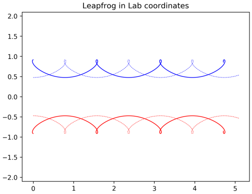
N	$h_c^{(N)}$
1	0.154700538379256
2	0.125362196172840
3	0.125302181592097
4	0.125039391697053
⋮	⋮
20	0.125000000000009

The fundamental question

How do new nonlinear behaviors emerge as the parameters change?
As α is decreased from 1 (as h is increased from 0) how do leapfrogging and escape begin to appear?

Reference: Under preparation

Aref coordinates work poorly for walkabout



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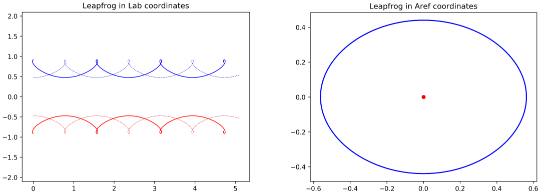

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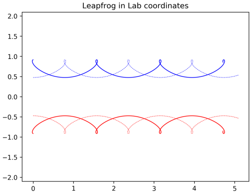

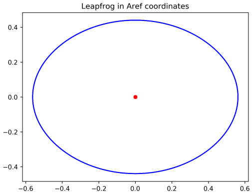

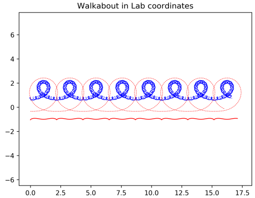

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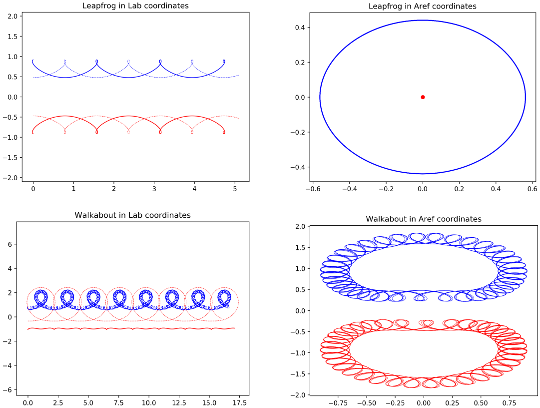

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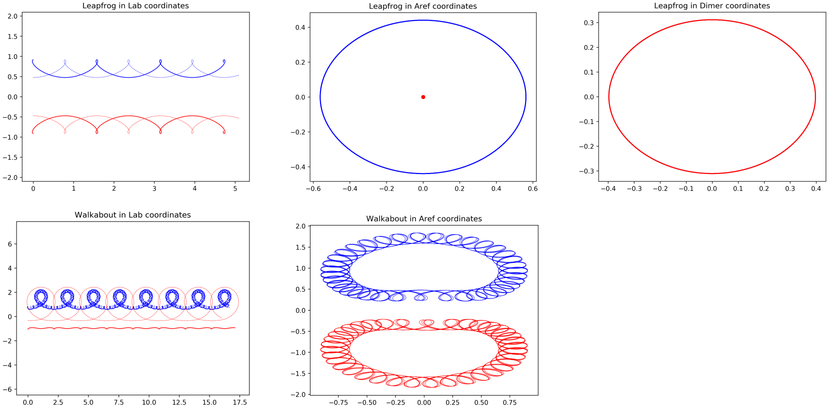

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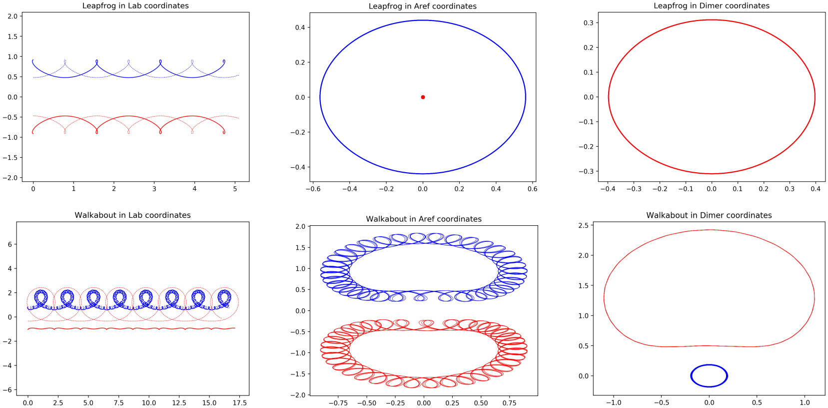

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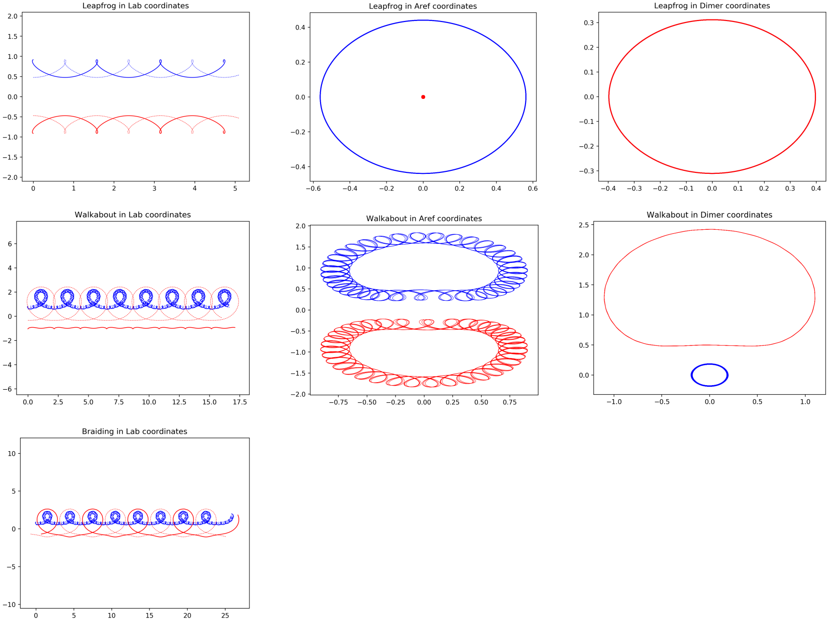

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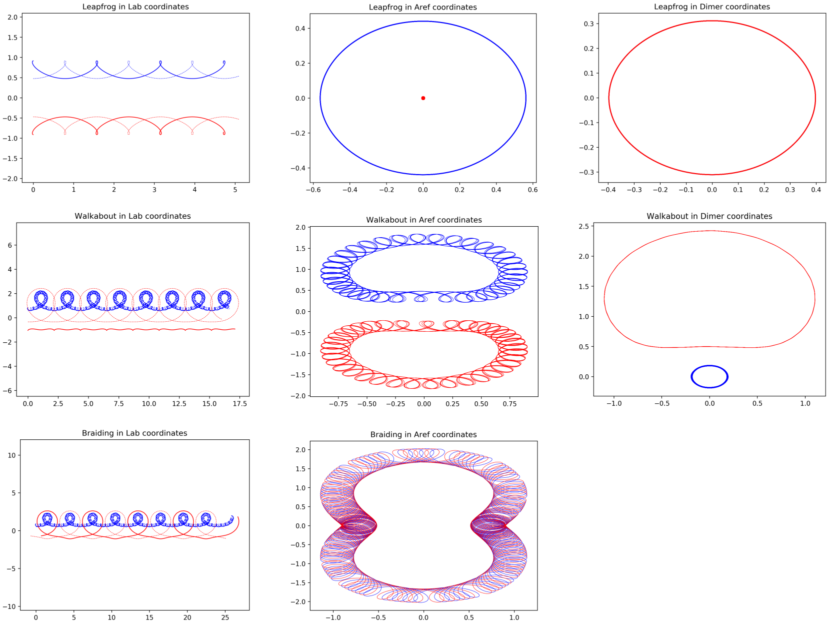

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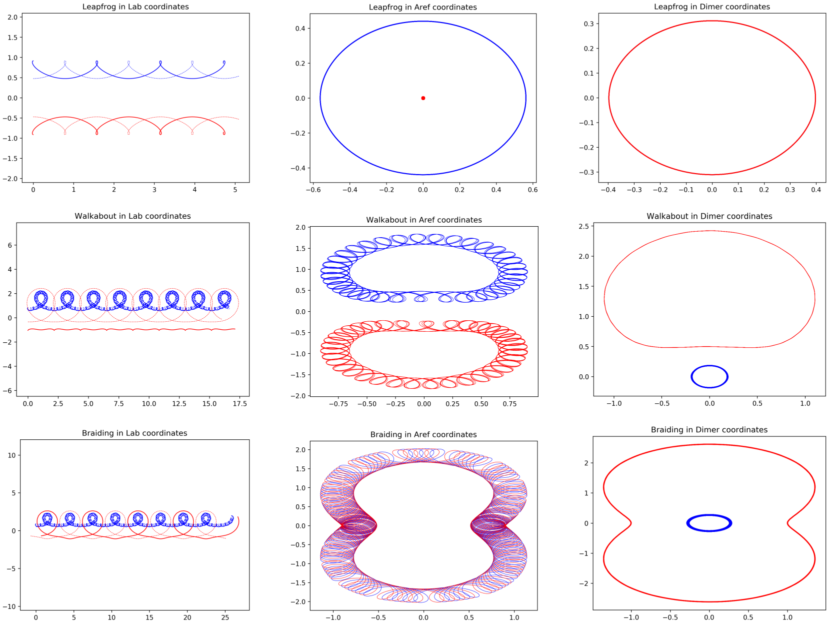
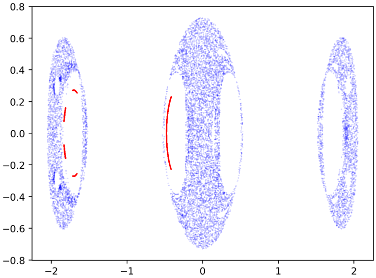


Poincaré surfaces of section

In Aref coordinates, orbits live on several disconnected pieces.



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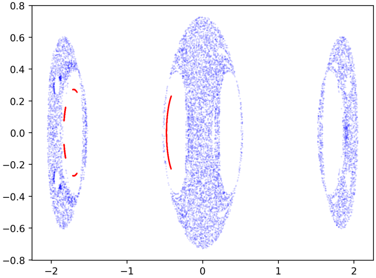
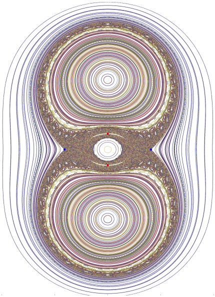


In dimer coordinates, orbits structure more logical and. . .



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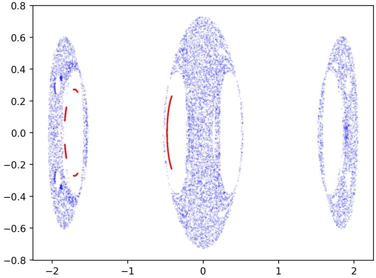

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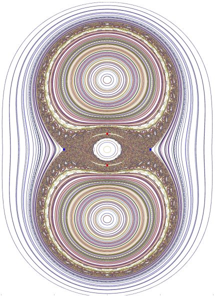
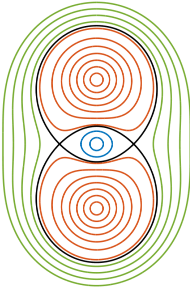


. . . resembles a reduced 3-vortex dynamics with vorticities $1 : 1 : -2$ studied by Rott and Aref.



A better coordinate system

After some work, we (*i.e.*, Brandon) rewrite the system in Dimer Coordinates in a new form

$$H(\zeta_-, \zeta_+) = H_{01}(\zeta_-) + H_{02}(\zeta_+) + H_1(\zeta_-, \zeta_+),$$

with coordinates

$$\zeta_- = z_1^- - z_2^-,\ \zeta_+ = z_1^+ - z_2^+,\ z_- = \frac{1}{2} z_1^- + z_2^-,\ z_+ = \frac{1}{2} z_1^+ + z_2^+,\ M = z_+ - z_-,$$

and where

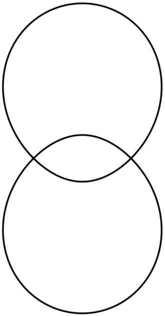
$$H_{01}(\zeta_-) = -\log|\zeta_-|^2, \text{ Nonlinear phase oscillator}$$

$$H_{02}(\zeta_+) = -\left(\log|\zeta_+|^2 - 2\log|\zeta_+ + M|^2 - 2\log|\zeta_+ - M|\right), \text{ Rott-Aref Hamiltonian}$$

$$H_1(\zeta_+, \zeta_-) = \log\frac{|\zeta_+ - \zeta_- - M|^2}{|\zeta_+ - M|^2} + \log\frac{|\zeta_+ + \zeta_- - M|^2}{|\zeta_+ - M|^2} + \log\frac{|\zeta_+ - \zeta_- + M|^2}{|\zeta_+ + M|^2} + \log\frac{|\zeta_+ + \zeta_- + M|^2}{|\zeta_+ + M|^2}. \text{ Coupling Term}$$

The Reduced $1 : 1 : -2$ System (Rott-Aref 1989)

Phase plane



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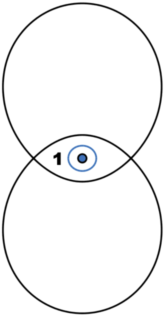
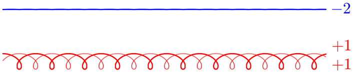


Region 1:
"Leapfrog"



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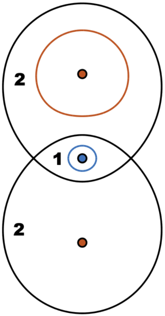

Region 1:
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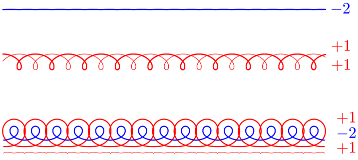
Region 2:
"Walkabout"



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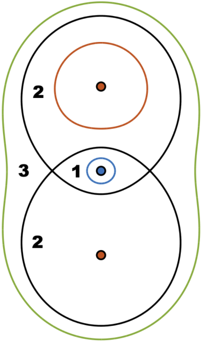

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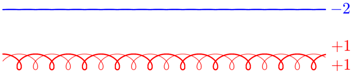

Region 2:
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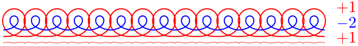
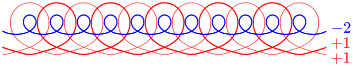


Region 3:
“Braiding”



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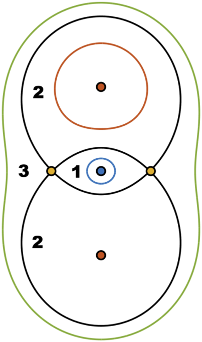

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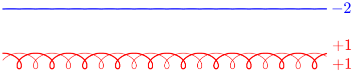

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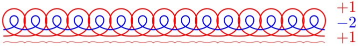

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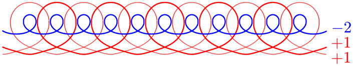
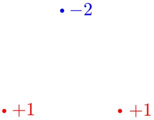


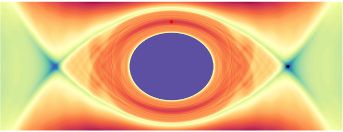
Saddles:
Rigidly translating equilateral triangles



How Escape Happens (Preliminary)

The (newish) technology of Lagrangian Descriptors allows visualization of invariant manifold without explicitly computing them.

$h = 0.12$



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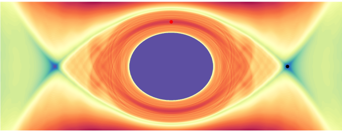

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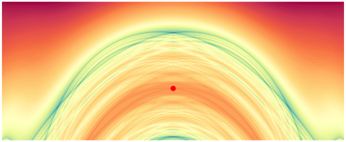

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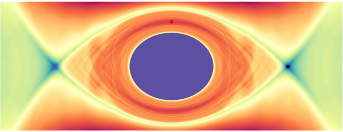
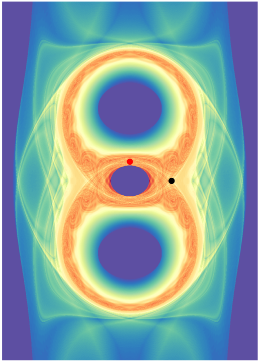


$h = 0.19$



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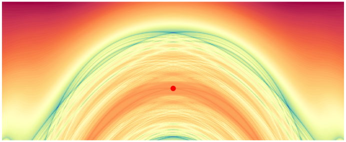

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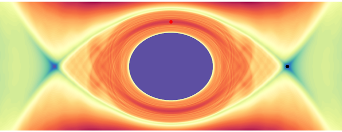

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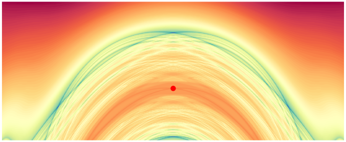

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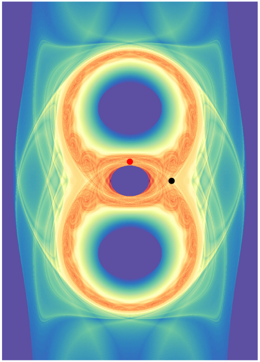
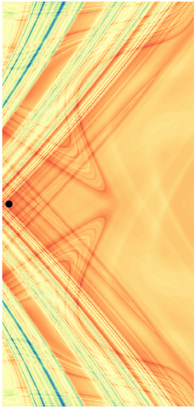


$h = 0.2$



Future directions

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- Leapfrogging with nonidentical pairs $\Gamma_1^- = -\Gamma_1^+$ and $\Gamma_2^- = -\Gamma_2^+$
- Leapfrogging orbits with $n \geq 3$ $(+1, -1)$ pairs.
- Leapfrogging on a sphere.
- Closely connected problem of scattering of vortex dipoles

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Thanks! Suggestions to `goodman@njit.edu` are very welcome.